Compact Manifolds with Exceptional Holonomy

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Abstract. In the classification of Riemannian holonomy groups, the *exceptional holonomy groups* are G_2 in 7 dimensions, and $Spin(7)$ in 8 dimensions. We outline the construction of the first known examples of compact 7- and 8-manifolds with holonomy G_2 and $Spin(7)$.

In the case of G_2, we first choose a finite group Γ of automorphisms of the torus T^7 and a flat Γ-invariant G_2-structure on T^7, so that T^7/Γ is an *orbifold*. Then we resolve the singularities of T^7/Γ to get a compact 7-manifold M. Finally we use analysis, and an understanding of Calabi-Yau metrics, to construct a family of metrics with holonomy G_2 on M, which converge to the singular metric on T^7/Γ.

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In the theory of Riemannian holonomy groups, perhaps the most mysterious are the two exceptional cases, the holonomy group G_2 in 7 dimensions and the holonomy group $Spin(7)$ in 8 dimensions. We shall describe the construction of the first known examples of *compact* 7-manifolds with holonomy G_2. There is a very similar construction for compact 8-manifolds with holonomy $Spin(7)$, which we will not discuss because of lack of space. All the details can be found in the author's papers [5], [6], [7] and the forthcoming book [8]. A good reference on Riemannian holonomy groups, and G_2 and $Spin(7)$ in particular, is the book by Salamon [13].

1 Riemannian holonomy groups

Let M be a connected n-dimensional manifold, let g be a Riemannian metric on M, and let ∇ be the Levi-Civita connection of g. Let x, y be points in M joined by a smooth path γ. Then *parallel transport* along γ using ∇ defines an isometry between the tangent spaces T_xM, T_yM at x and y.

Definition 1.1 The *holonomy group* $\mathrm{Hol}(g)$ of g is the group of isometries of T_xM generated by parallel transport around closed loops based at x in M. We consider $\mathrm{Hol}(g)$ to be a subgroup of $O(n)$, defined up to conjugation by elements of $O(n)$. Then $\mathrm{Hol}(g)$ is independent of the base point x in M.

The classification of holonomy groups was achieved by Berger [1] in 1955.

THEOREM 1.2 *Let M be a simply-connected, n-dimensional manifold, and g an irreducible, nonsymmetric Riemannian metric on M. Then either*

(*i*) $\mathrm{Hol}(g) = SO(n)$,
(*ii*) $n = 2m$ *and* $\mathrm{Hol}(g) = SU(m)$ *or* $U(m)$,
(*iii*) $n = 4m$ *and* $\mathrm{Hol}(g) = Sp(m)$ *or* $Sp(m)Sp(1)$,
(*iv*) $n = 7$ *and* $\mathrm{Hol}(g) = G_2$, *or*
(*v*) $n = 8$ *and* $\mathrm{Hol}(g) = Spin(7)$.

Now G_2 and $Spin(7)$ are the exceptional cases in this classification, so they are called the *exceptional holonomy groups*. For some time after Berger's classification, the exceptional holonomy groups remained a mystery. In 1987, Bryant [2] used the theory of exterior differential systems to show that locally there exist many metrics with these holonomy groups, and gave some explicit, incomplete examples. Then in 1989, Bryant and Salamon [3] found explicit, *complete* metrics with holonomy G_2 and $Spin(7)$ on noncompact manifolds. In 1994-5 the author constructed examples of metrics with holonomy G_2 and $Spin(7)$ on *compact* manifolds [5, 6, 7, 8], and these are the subject of this article.

We now introduce the holonomy group G_2. Let $(x_1, \ldots, x_7)$ be coordinates on $\mathbb{R}^7$. Define a metric g_0 and a 3-form φ_0 on $\mathbb{R}^7$ by

$$g_0 = dx_1^2 + \cdots + dx_7^2, \tag{1}$$

$$\begin{aligned}\varphi_0 = {} & dx_1 \wedge dx_2 \wedge dx_7 + dx_1 \wedge dx_3 \wedge dx_6 + dx_1 \wedge dx_4 \wedge dx_5 + dx_2 \wedge dx_3 \wedge dx_5 \\ & - dx_2 \wedge dx_4 \wedge dx_6 + dx_3 \wedge dx_4 \wedge dx_7 + dx_5 \wedge dx_6 \wedge dx_7.\end{aligned} \tag{2}$$

The subgroup of $GL(7, \mathbb{R})$ preserving φ_0 is the *exceptional Lie group* G_2. This group also preserves g_0 and the orientation on $\mathbb{R}^7$. It is a compact, semisimple, 14-dimensional Lie group, a subgroup of $SO(7)$.

A *G_2-structure* on a 7-manifold M is a principal subbundle of the frame bundle of M, with structure group G_2. Each G_2-structure gives rise to a 3-form φ and a metric g on M, such that every tangent space of M admits an isomorphism with $\mathbb{R}^7$ identifying φ and g with φ_0 and g_0 respectively. By an abuse of notation, we will refer to (φ, g) as a G_2-structure.

PROPOSITION 1.3 *Let M be a 7-manifold and (φ, g) a G_2-structure on M. Then the following are equivalent:*

(*i*) $\mathrm{Hol}(g) \subseteq G_2$, *and φ is the induced 3-form,*
(*ii*) $\nabla\varphi = 0$ *on M, where ∇ is the Levi-Civita connection of g, and*
(*iii*) $d\varphi = d^*\varphi = 0$ *on M.*

We call $\nabla\varphi$ the *torsion* of the G_2-structure (φ, g), and when $\nabla\varphi = 0$ the G_2-structure is *torsion-free*. If (φ, g) is torsion-free, then g is Ricci-flat.

PROPOSITION 1.4 *Let M be a compact 7-manifold, and suppose that (φ, g) is a torsion-free G_2-structure on M. Then $\mathrm{Hol}(g) = G_2$ if and only if $\pi_1(M)$ is finite. In this case the moduli space of metrics with holonomy G_2 on M, up to diffeomorphisms isotopic to the identity, is a smooth manifold of dimension $b^3(M)$.*

2 A 'Kummer construction' for a 7-manifold

It is well known that metrics with holonomy $SU(2)$ on the $K3$ surface can be obtained by resolving the 16 singularities of the orbifold $T^4/\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts on T^4 with 16 fixed points. This is called the *Kummer construction*. Our construction is motivated by and modelled on this. It can be divided into four steps. Here is a summary of each. For simplicity we will describe the G_2 case only, but the $Spin(7)$ case is very similar.

Step 1. Let T^7 be the 7-torus. Let (φ_0, g_0) be a flat G_2-structure on T^7. Choose a finite group Γ of isometries of T^7 preserving (φ_0, g_0). Then the quotient T^7/Γ is a singular, compact 7-manifold.

Step 2. For certain special groups Γ there is a method to resolve the singularities of T^7/Γ in a natural way, using complex geometry. We get a nonsingular, compact 7-manifold M, together with a map $\pi : M \to T^7/\Gamma$, the resolving map.

Step 3. On M, we explicitly write down a 1-parameter family of G_2-structures (φ_t, g_t) depending on a real variable $t \in (0, \epsilon)$. These G_2-structures are not torsion-free, but when t is small, they have small torsion. As $t \to 0$, the G_2-structure (φ_t, g_t) converges to the singular G_2-structure $\pi^*(\varphi_0, g_0)$.

Step 4. We prove using analysis that for all sufficiently small t, the G_2-structure (φ_t, g_t) on M, with small torsion, can be deformed to a G_2-structure $(\tilde{\varphi}_t, \tilde{g}_t)$, with zero torsion. Finally, we show that $\tilde{g}_t$ is a metric with holonomy G_2 on the compact 7-manifold M.

We will now explain the steps in greater detail.

Step 1

Here is an example of a suitable group Γ. Let $(x_1, \ldots, x_7)$ be coordinates on $T^7 = \mathbb{R}^7/\mathbb{Z}^7$, where $x_i \in \mathbb{R}/\mathbb{Z}$. Let (φ_0, g_0) be the flat G_2-structure on T^7 defined by (2). Let α, β and γ be the involutions of T^7 defined by

$$\alpha\big((x_1, \ldots, x_7)\big) = \big(-x_1, -x_2, -x_3, -x_4, x_5, x_6, x_7\big), \tag{3}$$

$$\beta\big((x_1, \ldots, x_7)\big) = \big(-x_1, \tfrac{1}{2} - x_2, x_3, x_4, -x_5, -x_6, x_7\big), \tag{4}$$

$$\gamma\big((x_1, \ldots, x_7)\big) = \big(\tfrac{1}{2} - x_1, x_2, \tfrac{1}{2} - x_3, x_4, -x_5, x_6, -x_7\big). \tag{5}$$

By inspection, α, β and γ preserve (φ_0, g_0), because of the careful choice of exactly which signs to change. Also, $\alpha^2 = \beta^2 = \gamma^2 = 1$, and α, β and γ commute. Thus they generate a group $\Gamma = \langle \alpha, \beta, \gamma \rangle \cong \mathbb{Z}_2^3$ of isometries of T^7 preserving the flat G_2-structure (φ_0, g_0).

Lemma 2.1 *The elements $\beta\gamma$, $\gamma\alpha$, $\alpha\beta$ and $\alpha\beta\gamma$ of Γ have no fixed points on T^7. The fixed points of α, β, γ are each 16 copies of T^3. The singular set S of T^7/Γ is a disjoint union of 12 copies of T^3, 4 copies from each of α, β, γ. Each component of S is a singularity modelled on that of $T^3 \times \mathbb{C}^2/\{\pm 1\}$.*

Thus the singular set splits into a disjoint union of connected components, and each component is very simple. This is helpful because we can desingularize each connected component independently, and simple singularities are easier to resolve.

STEP 2

Our goal is to resolve the singular set S of T^7/Γ to get a compact 7-manifold M with holonomy G_2. How can we do this? In general we cannot, because we have no idea of how to resolve general orbifold singularities with holonomy G_2. However, suppose we can arrange that every connected component of S is locally isomorphic to either

(*a*) $T^3 \times \mathbb{C}^2/G$, for G a finite subgroup of $SU(2)$, or
(*b*) $\mathcal{S}^1 \times \mathbb{C}^3/G$, for G a finite subgroup of $SU(3)$ acting freely on $\mathbb{C}^3 \setminus 0$.

In this case we can use *complex algebraic geometry* to find a natural resolution X of $\mathbb{C}^2/G$ or Y of $\mathbb{C}^3/G$, and then $T^3 \times X$ or $\mathcal{S}^1 \times Y$ gives a local model for how to resolve the corresponding component of S in T^7/Γ.

In case (*a*), X must have a Kähler metric h with holonomy $SU(2)$ that is asymptotic to the flat Euclidean metric on $\mathbb{C}^2/G$. Such metrics are called *Asymptotically Locally Euclidean* (ALE). They have been classified by Kronheimer [10, 11], and they exist for every finite subgroup $G \subset SU(2)$. The point is that if X has holonomy $SU(2)$, then the product 7-manifold $T^3 \times X$ has holonomy $\{1\} \times SU(2)$. But $\{1\} \times SU(2)$ is a subgroup of G_2, and so $T^3 \times X$ has a torsion-free G_2-structure by Proposition 1.3. Hence, $T^3 \times X$ gives a local model for how to resolve the singularity $T^3 \times \mathbb{C}^2/G$ with holonomy G_2.

In case (*b*), Y is a *crepant resolution* of $\mathbb{C}^3/G$, and carries an ALE Kähler metric h with holonomy $SU(3)$. Such resolutions and metrics exist for all finite $G \subset SU(3)$, by work of Roan [12] and the author [8]. Since $\{1\} \times SU(3) \subset G_2$, if (Y, h) has holonomy $SU(3)$ then $\mathcal{S}^1 \times Y$ has a torsion-free G_2-structure, and provides a local model for how to resolve the singularity $\mathcal{S}^1 \times \mathbb{C}^3/G$ with holonomy G_2.

Suppose that all the singularities of T^7/Γ are of type (*a*) or (*b*). Then we can construct a compact, nonsingular 7-manifold M by resolving each singularity $T^3 \times \mathbb{C}^2/G$ using $T^3 \times X$, and resolving each singularity $\mathcal{S}^1 \times \mathbb{C}^3/G$ using $\mathcal{S}^1 \times Y$, as above. In the example this means gluing 12 copies of $T^3 \times X$ into T^7/Γ, where X is the blow-up of $\mathbb{C}^2/\{\pm 1\}$ at its singular point.

STEP 3

For each resolution X of $\mathbb{C}^2/G$ in case (*a*), and Y of $\mathbb{C}^3/G$ in case (*b*), we can find a 1-parameter family $\{h_t : t > 0\}$ of metrics with the properties

(*a*) h_t is a Kähler metric on X with $\mathrm{Hol}(h_t) = SU(2)$. Its injectivity radius satisfies $\delta(h_t) = O(t)$, its Riemann curvature satisfies $\|R(h_t)\|_{C^0} = O(t^{-2})$, and $h_t = h + O(t^4 r^{-4})$ for large r, where h is the Euclidean metric on $\mathbb{C}^2/G$, and r the distance from the origin.

(*b*) h_t is Kähler on Y with $\mathrm{Hol}(h_t) = SU(3)$, satisfying $\delta(h_t) = O(t)$, $\|R(h_t)\|_{C^0} = O(t^{-2})$, and $h_t = h + O(t^6 r^{-6})$ for large r.

In fact we can choose h_t to be isometric to $t^2 h_1$, and the properties above are easy to prove.

Suppose one of the components of the singular set S of T^7/Γ is locally modelled on $T^3 \times \mathbb{C}^2/G$. Then T^3 has a natural flat metric h_{T^3}. Let X be the resolution of $\mathbb{C}^2/G$ and let $\{h_t : t > 0\}$ satisfy property (a). Then $\hat{g}_t = h_{T^3} + h_t$ is a metric on $T^3 \times X$ with holonomy $\{1\} \times SU(2)$, which is contained in G_2. Thus there is an associated torsion-free G_2-structure $(\hat{\varphi}_t, \hat{g}_t)$ on $T^3 \times X$. Similarly, if a component of S is modelled on $\mathcal{S}^1 \times \mathbb{C}^3/G$, we get a family of torsion-free G_2-structures $(\hat{\varphi}_t, \hat{g}_t)$ on $\mathcal{S}^1 \times Y$.

The idea is to make a G_2-structure (φ_t, g_t) on M by gluing together the torsion-free G_2-structures $(\hat{\varphi}_t, \hat{g}_t)$ on the patches $T^3 \times X$ and $\mathcal{S}^1 \times Y$, and (φ_0, g_0) on T^7/Γ. The gluing is done using a partition of unity. Naturally, the first derivative of the partition of unity introduces 'errors', so that (φ_t, g_t) is not torsion-free. The size of the torsion $\nabla\varphi_t$ depends on the difference $\hat{\varphi}_t - \varphi_0$ in the region where the partition of unity changes. On the patches $T^3 \times X$, since $h_t - h = O(t^4 r^{-4})$ and the partition of unity has nonzero derivative when $r = O(1)$, we find that $\nabla\varphi_t = O(t^4)$. Similarly $\nabla\varphi_t = O(t^6)$ on the patches $\mathcal{S}^1 \times Y$, and so $\nabla\varphi_t = O(t^4)$ on M.

For small t, the dominant contributions to the injectivity radius $\delta(g_t)$ and Riemann curvature $R(g_t)$ are made by those of the metrics h_t on X and Y, so we expect $\delta(g_t) = O(t)$ and $\|R(g_t)\|_{C^0} = O(t^{-2})$ by properties (a) and (b) above. In this way we prove the following result, which gives the estimates on (φ_t, g_t) that we need.

Theorem A *On the compact 7-manifold M described above, and on many other 7-manifolds constructed in a similar fashion, one can write down the following data explicitly in coordinates:*

- *Positive constants A_1, A_2, A_3 and ϵ,*
- *A G_2-structure (φ_t, g_t) on M with $d\varphi_t = 0$ for each $t \in (0, \epsilon)$, and*
- *A 3-form ψ_t on M with $d^*\psi_t = d^*\varphi_t$ for each $t \in (0, \epsilon)$.*

These satisfy three conditions:

(*i*) $\|\psi_t\|_{L^2} \leq A_1 t^4$ *and* $\|d^*\psi_t\|_{L^{14}} \leq A_1 t^4$,

(*ii*) *the injectivity radius $\delta(g_t)$ satisfies $\delta(g_t) \geq A_2 t$,*

(*iii*) *the Riemann curvature $R(g_t)$ of g_t satisfies $\|R(g_t)\|_{C^0} \leq A_3 t^{-2}$.*

Here the operator d^* and the norms $\|\,.\,\|_{L^2}$, $\|\,.\,\|_{L^{14}}$ and $\|\,.\,\|_{C^0}$ depend on g_t.

Here one should regard ψ_t as a *first integral* of the torsion $\nabla\varphi_t$ of (φ_t, g_t). Thus the norms $\|\psi_t\|_{L^2} \leq A_1 t^4$ and $\|d^*\psi_t\|_{L^{14}} \leq A_1 t^4$ are measures of $\nabla\varphi_t$. So parts (i)-(iii) say that the torsion $\nabla\varphi_t$ must be small compared to the injectivity radius and Riemann curvature of (M, g_t).

STEP 4

We prove the following analysis result.

THEOREM B *In the situation of Theorem A there are constants $\kappa, K > 0$ depending only on A_1, A_2, A_3 and ϵ, such that for each $t \in (0, \kappa]$ there exists a smooth, torsion-free G_2-structure $(\tilde\varphi_t, \tilde g_t)$ on M with $\|\tilde\varphi_t - \varphi_t\|_{C^0} \le K t^{1/2}$.*

Basically, this result says that if (φ, g) is a G_2-structure on M, and the torsion $\nabla\varphi$ is sufficiently small, then we can deform to a nearby G_2-structure $(\tilde\varphi, \tilde g)$ that is torsion-free. Here is a sketch of the proof of Theorem B, ignoring several technical points. The proof is that given in [8], which is an improved version of the proof in [5]. For simplicity we omit the subscripts t.

We have a 3-form φ with $d\varphi = 0$ and $d^*\varphi = d^*\psi$ for small ψ, and we wish to construct a nearby 3-form $\tilde\varphi$ with $d\tilde\varphi = 0$ and $\tilde d^*\tilde\varphi = 0$. Set $\tilde\varphi = \varphi + d\eta$, where η is a small 2-form. Then η must satisfy a nonlinear p.d.e., which we write as

$$d^*d\eta = -d^*\psi + d^*F(d\eta), \tag{6}$$

where F is nonlinear, satisfying $F(d\eta) = O\big(|d\eta|^2\big)$.

We solve (6) by iteration, introducing a sequence $\{\eta_j\}_{j=0}^\infty$ with $\eta_0 = 0$, satisfying the inductive equations

$$d^*d\eta_{j+1} = -d^*\psi + d^*F(d\eta_j), \qquad d^*\eta_{j+1} = 0. \tag{7}$$

If such a sequence exists and converges to η, then taking the limit in (7) shows that η satisfies (6), giving us the solution we want.

The key to proving this is an *inductive estimate* on the sequence $\{\eta_j\}_{j=0}^\infty$. The inductive estimate we use has three ingredients, the equations

$$\|d\eta_{j+1}\|_{L^2} \le \|\psi\|_{L^2} + C_1\|d\eta_j\|_{L^2}\|d\eta_j\|_{C^0}, \tag{8}$$

$$\|\nabla d\eta_{j+1}\|_{L^{14}} \le C_2\big(\|d^*\psi\|_{L^{14}} + \|\nabla d\eta_j\|_{L^{14}}\|d\eta_j\|_{C^0} + t^{-4}\|d\eta_{j+1}\|_{L^2}\big), \tag{9}$$

$$\|d\eta_j\|_{C^0} \le C_3\big(t^{1/2}\|\nabla d\eta_j\|_{L^{14}} + t^{-7/2}\|d\eta_j\|_{L^2}\big). \tag{10}$$

Here C_1, C_2, C_3 are positive constants independent of t. Equation (8) is obtained from (7) by taking the L^2-inner product with η_{j+1} and integrating by parts. Using the fact that $d^*\varphi = d^*\psi$ and ψ is $O(t^4)$, we get a powerful a priori estimate of the L^2-norm of $d\eta_{j+1}$.

Equation (9) is derived from an *elliptic regularity estimate* for the operator $d + d^*$ acting on 3-forms on M. Equation (10) follows from the *Sobolev embedding theorem*, since $L^{14}_1(M)$ embeds in $C^0(M)$. Both (9) and (10) are proved on small balls of radius $O(t)$ in M, using parts (*ii*) and (*iii*) of Theorem A, and this is where the powers of t come from.

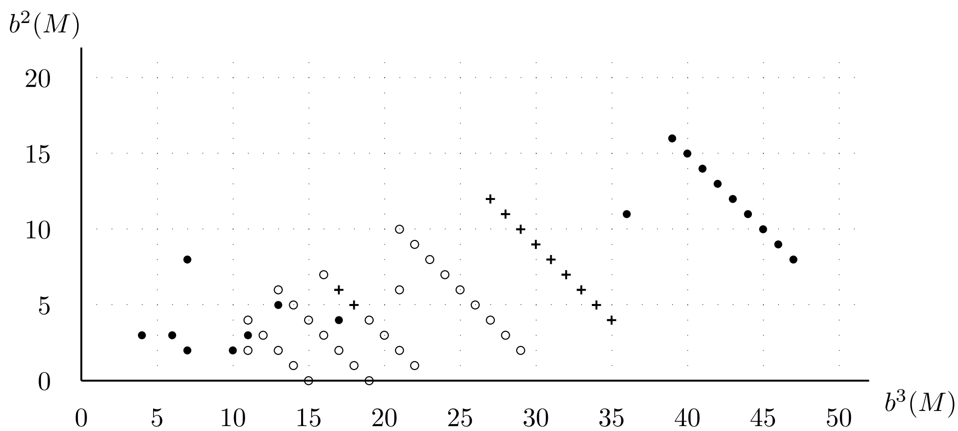
Using (8)-(10) and part (*i*) of Theorem A we show that if

$$\|d\eta_j\|_{L^2} \le C_4 t^4, \quad \|\nabla d\eta_j\|_{L^{14}} \le C_5, \quad \text{and} \quad \|d\eta_j\|_{C^0} \le K t^{1/2}, \tag{11}$$

where C_4, C_5 and K are positive constants depending on C_1, C_2, C_3 and A_1, and if t is sufficiently small, then the same inequalities (11) apply to $d\eta_{j+1}$. Since $\eta_0 = 0$, by induction (11) applies for all j and the sequence $\{d\eta_j\}_{j=0}^{\infty}$ is bounded in the Banach space $L_1^{14}(\Lambda^3 T^*M)$. One can then use standard techniques in analysis to prove that this sequence converges to a smooth limit $d\eta$. This concludes the sketch proof of Theorem B.

From Theorems A and B we see that the compact 7-manifold M constructed in Step 2 admits torsion-free G_2-structures $(\tilde{\varphi}, \tilde{g})$. Proposition 1.4 then shows that $\mathrm{Hol}(\tilde{g}) = G_2$ if and only if $\pi_1(M)$ is finite. In the example above M is simply-connected, and so $\pi_1(M) = \{1\}$ and M has metrics with holonomy G_2, as we want.

By considering different groups Γ acting on T^7, and also by finding topologically distinct resolutions $M_1, \ldots, M_k$ of the same orbifold T^7/Γ, we can construct many compact Riemannian 7-manifolds with holonomy G_2. Here is a graph of the Betti numbers $b^2(M)$ and $b^3(M)$ of the 68 examples found in [5, 6]. More examples will be given in [8].



Betti numbers of known compact 7-manifolds with holonomy G_2

On this graph the symbol '$\bullet$' denotes the Betti numbers of a simply-connected 7-manifold, '$\circ$' denotes a non-simply-connected manifold, and '+' denotes both a simply-connected and a non-simply-connected manifold.

So far we have discussed only the holonomy group G_2. There is a very similar construction for compact manifolds with holonomy $Spin(7)$, described in [7] and [8]. Here are some of the similarities and differences in the two cases. The holonomy group $Spin(7)$ is a subgroup of $SO(8)$, a compact 21-dimensional Lie group isomorphic to the double cover of $SO(7)$. It is the subgroup of $GL(8, \mathbb{R})$ preserving a certain 4-form Ω_0 on $\mathbb{R}^8$, and also preserves the Euclidean metric g_0 on $\mathbb{R}^8$.

Thus a $Spin(7)$-structure on an 8-manifold M is equivalent to a pair (Ω, g), where Ω is a 4-form and g a Riemann metric that are pointwise isomorphic to Ω_0 and g_0. Riemannian manifolds with holonomy $Spin(7)$ are Ricci-flat. Compact manifolds M with holonomy $Spin(7)$ are simply-connected spin manifolds, and computing the index of the Dirac operator shows that their Betti numbers must satisfy $b^3(M) + b^4_+(M) = b^2(M) + b^4_-(M) + 25$.

We can construct compact 8-manifolds with holonomy $Spin(7)$ by resolving the singularities of orbifolds T^8/Γ. The construction is more difficult than the G_2 case in two ways. Firstly, it seems to be more difficult to find suitable orbifolds T^8/Γ, and it is necessary to consider more complicated kinds of orbifold singularities. Secondly, the analysis is more difficult, and one has to try harder to make the sequence converge. In [7] we find at least 95 topologically distinct compact 8-manifolds with holonomy $Spin(7)$, realizing 29 distinct sets of Betti numbers, and [8] will give more examples.

Note that compact manifolds with holonomy G_2 and $Spin(7)$ are examples of compact Ricci-flat Riemannian manifolds. In fact, compact manifolds with holonomy G_2 are the only known source of odd-dimensional examples of compact, simply-connected Ricci-flat Riemannian manifolds.

3 Directions for future research

Here are four areas in which I hope to see interesting developments soon.

- *Other constructions of compact manifolds with exceptional holonomy.* The author has extended the constructions of [5]-[7] to include resolutions of more general quotient singularities, in particular *non-isolated* quotient singularities $\mathbb{C}^m/G$ for G a finite subgroup of $SU(m)$ and $m = 3$ or 4, and the results will be published in [8]. Another promising possibility is to try to replace the orbifold T^7/Γ by $(\mathcal{S}^1 \times W)/\Gamma$, where W is a Calabi-Yau 3-fold.

- Harvey and Lawson's theory of *calibrated geometry* [4] singles out three classes of special submanifolds in manifolds of exceptional holonomy: *associative 3-folds* and *coassociative 4-folds* in G_2-manifolds, and *Cayley 4-folds* in $Spin(7)$-manifolds. They are minimal submanifolds, and have good properties under deformation. Compact examples can be constructed as the fixed point sets of isometries, as in [6].

  It would be interesting to study families of compact manifolds of these types, to understand the way singularities develop in such families, and whether a compact G_2 or $Spin(7)$-manifold can be fibred by coassociative or Cayley 4-manifolds, with some singular fibres.

- *Gauge theory on compact* $Spin(7)$*-manifolds.* Let M be a compact 8-manifold with holonomy $Spin(7)$, let E be a vector bundle or principal bundle over M, and let A be a connection on E. Then the curvature F_A of A is a 2-form with values in $\mathrm{ad}(E)$. Now the $Spin(7)$-structure induces a splitting $\Lambda^2 T^*M = \Lambda^2_7 \oplus \Lambda^2_{21}$, where $\Lambda^2_7, \Lambda^2_{21}$ are vector bundles over M with fibre $\mathbb{R}^7$, $\mathbb{R}^{21}$ respectively. We call A a $Spin(7)$*-instanton* if the component of F_A in $\mathrm{ad}(E) \otimes \Lambda^2_7$ is zero.

  It turns out that $Spin(7)$-instantons have many properties in common with instantons in 4 dimensions, that are studied in Donaldson theory. Christopher Lewis and the author [9] have proved an existence theorem for $Spin(7)$-instantons with gauge group $SU(2)$ on certain compact 8-manifolds with holonomy $Spin(7)$. In 4 dimensions a sequence of instantons can 'bubble' at a finite number of points. In 8 dimensions we expect 'bubbling' to occur instead around a compact Cayley 4-manifold, and we construct families of instantons in which this happens.

- *Connections with String Theory.* String Theory is a branch of high-energy theoretical physics that aims to unify quantum theory and gravity by modelling particles as 1-dimensional objects called *strings*. One of its features is that it prescribes the dimension of space-time. This depends on the details of the theory, but the most popular model, *supersymmetric string theory*, gives dimension 10. To explain the discrepancy between this and the 4 space-time dimensions that we observe, it is supposed that the universe looks locally like $\mathbb{R}^4 \times M^6$, where M^6 is a compact 6-manifold with very small radius, of order 10^{-33}cm.

  In supersymmetric string theory, M must be a *Calabi-Yau 3-fold.* So string theorists are interested in Calabi-Yau 3-folds, and have contributed many ideas to the subject, including that of *Mirror Symmetry.* However, if instead we consider $\mathbb{R}^3 \times M^7$, corresponding to an observable universe with 3 space-time dimensions, then by work of Vafa and Shatashvili M^7 must be a compact 7-manifold with holonomy G_2. Similarly, if we consider $\mathbb{R}^2 \times M^8$, so that the observable universe has 2 space-time dimensions, then M^8 is a compact 8-manifold with holonomy $Spin(7)$.

  Recently, string theorists have begun to seriously consider the possibility that the universe may have 11 dimensions ('M theory') or even 12 dimensions ('F theory'). To reduce to 4 observable space-time dimensions in these theories will require a manifold of dimension 7 or 8, and it seems likely that compact manifolds with exceptional holonomy will play a rôle in this.

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