Compact Manifolds with Exceptional Holonomy

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Abstract. In the classification of Riemannian holonomy groups, the exceptionnal holonomy groups are $G_2$ in 7 dimensions, and $\text{Spin}(7)$ in 8 dimensions. We outline the construction of the first known examples of compact 7- and 8-manifolds with holonomy $G_2$ and $\text{Spin}(7)$.

In the case of $G_2$, we first choose a finite group $\Gamma$ of automorphisms of the torus $T^7$ and a flat $\Gamma$-invariant $G_2$-structure on $T^7$, so that $T^7/\Gamma$ is an orbifold. Then we resolve the singularities of $T^7/\Gamma$ to get a compact 7-manifold $M$. Finally we use analysis, and an understanding of Calabi-Yau metrics, to construct a family of metrics with holonomy $G_2$ on $M$, which converge to the singular metric on $T^7/\Gamma$.

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In the theory of Riemannian holonomy groups, perhaps the most mysterious are the two exceptional cases, the holonomy group $G_2$ in 7 dimensions and the holonomy group $\text{Spin}(7)$ in 8 dimensions. We shall describe the construction of the first known examples of compact 7-manifolds with holonomy $G_2$. There is a very similar construction for compact 8-manifolds with holonomy $\text{Spin}(7)$, which we will not discuss because of lack of space. All the details can be found in the author’s papers [5], [6], [7] and the forthcoming book [8]. A good reference on Riemannian holonomy groups, and $G_2$ and $\text{Spin}(7)$ in particular, is the book by Salamon [13].

1 Riemannian Holonomy Groups

Let $M$ be a connected $n$-dimensional manifold, let $g$ be a Riemannian metric on $M$, and let $\nabla$ be the Levi-Civita connection of $g$. Let $x, y$ be points in $M$ joined by a smooth path $\gamma$. Then parallel transport along $\gamma$ using $\nabla$ defines an isometry between the tangent spaces $T_x M, T_y M$ at $x$ and $y$.

Definition 1.1 The holonomy group $\text{Hol}(g)$ of $g$ is the group of isometries of $T_x M$ generated by parallel transport around closed loops based at $x$ in $M$. We consider $\text{Hol}(g)$ to be a subgroup of $O(n)$, defined up to conjugation by elements of $O(n)$. Then $\text{Hol}(g)$ is independent of the base point $x$ in $M$.

The classification of holonomy groups was achieved by Berger [1] in 1955.
Theorem 1.2 Let $M$ be a simply-connected, $n$-dimensional manifold, and $g$ an irreducible, nonsymmetric Riemannian metric on $M$. Then either

(i) $\text{Hol}(g) = \text{SO}(n)$,
(ii) $n = 2m$ and $\text{Hol}(g) = \text{SU}(m)$ or $U(m)$,
(iii) $n = 4m$ and $\text{Hol}(g) = \text{Sp}(m)$ or $\text{Sp}(m)\text{Sp}(1)$,
(iv) $n = 7$ and $\text{Hol}(g) = G_2$, or
(v) $n = 8$ and $\text{Hol}(g) = \text{Spin}(7)$.

Now $G_2$ and $\text{Spin}(7)$ are the exceptional cases in this classification, so they are called the exceptional holonomy groups. For some time after Berger’s classification, the exceptional holonomy groups remained a mystery. In 1987, Bryant [2] used the theory of exterior differential systems to show that locally there exist many metrics with these holonomy groups, and gave some explicit, incomplete examples. Then in 1989, Bryant and Salamon [3] found explicit, complete metrics with holonomy $G_2$ and $\text{Spin}(7)$ on noncompact manifolds. In 1994-5 the author constructed examples of metrics with holonomy $G_2$ and $\text{Spin}(7)$ on compact manifolds [5, 6, 7, 8], and these are the subject of this article.

We now introduce the holonomy group $G_2$. Let $(x_1, \ldots, x_7)$ be coordinates on $\mathbb{R}^7$. Define a metric $g_0$ and a 3-form $\varphi_0$ on $\mathbb{R}^7$ by

\begin{align}
 g_0 &= dx_1^2 + \cdots + dx_7^2, \\
 \varphi_0 &= dx_1 \wedge dx_2 \wedge dx_7 + dx_1 \wedge dx_3 \wedge dx_6 + dx_1 \wedge dx_4 \wedge dx_5 + dx_2 \wedge dx_3 \wedge dx_5 \\
 &\quad - dx_2 \wedge dx_4 \wedge dx_6 + dx_3 \wedge dx_4 \wedge dx_7 + dx_5 \wedge dx_6 \wedge dx_7.
\end{align}

The subgroup of $\text{GL}(7, \mathbb{R})$ preserving $\varphi_0$ is the exceptional Lie group $G_2$. This group also preserves $g_0$ and the orientation on $\mathbb{R}^7$. It is a compact, semisimple, 14-dimensional Lie group, a subgroup of $\text{SO}(7)$.

A $G_2$-structure on a 7-manifold $M$ is a principal subbundle of the frame bundle of $M$, with structure group $G_2$. Each $G_2$-structure gives rise to a 3-form $\varphi$ and a metric $g$ on $M$, such that every tangent space of $M$ admits an isomorphism with $\mathbb{R}^7$ identifying $\varphi$ and $g$ with $\varphi_0$ and $g_0$ respectively. By an abuse of notation, we will refer to $(\varphi, g)$ as a $G_2$-structure.

Proposition 1.3 Let $M$ be a 7-manifold and $(\varphi, g)$ a $G_2$-structure on $M$. Then the following are equivalent:

(i) $\text{Hol}(g) \subseteq G_2$, and $\varphi$ is the induced 3-form,
(ii) $\nabla \varphi = 0$ on $M$, where $\nabla$ is the Levi-Civita connection of $g$, and
(iii) $d\varphi = d^* \varphi = 0$ on $M$.

We call $\nabla \varphi$ the torsion of the $G_2$-structure $(\varphi, g)$, and when $\nabla \varphi = 0$ the $G_2$-structure is torsion-free. If $(\varphi, g)$ is torsion-free, then $g$ is Ricci-flat.

Proposition 1.4 Let $M$ be a compact 7-manifold, and suppose that $(\varphi, g)$ is a torsion-free $G_2$-structure on $M$. Then $\text{Hol}(g) = G_2$ if and only if $\pi_1(M)$ is finite. In this case the moduli space of metrics with holonomy $G_2$ on $M$, up to diffeomorphisms isotopic to the identity, is a smooth manifold of dimension $b^3(M)$. 
2 A ‘Kummer construction’ for a 7-manifold

It is well known that metrics with holonomy $SU(2)$ on the $K3$ surface can be obtained by resolving the 16 singularities of the orbifold $T^4/\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts on $T^4$ with 16 fixed points. This is called the Kummer construction. Our construction is motivated by and modelled on this. It can be divided into four steps. Here is a summary of each. For simplicity we will describe the $G_2$ case only, but the $Spin(7)$ case is very similar.

Step 1. Let $T^7$ be the 7-torus. Let $(\varphi_0, g_0)$ be a flat $G_2$-structure on $T^7$. Choose a finite group $\Gamma$ of isometries of $T^7$ preserving $(\varphi_0, g_0)$. Then the quotient $T^7/\Gamma$ is a singular, compact 7-manifold.

Step 2. For certain special groups $\Gamma$ there is a method to resolve the singularities of $T^7/\Gamma$ in a natural way, using complex geometry. We get a non-singular, compact 7-manifold $M$, together with a map $\pi : M \to T^7/\Gamma$, the resolving map.

Step 3. On $M$, we explicitly write down a 1-parameter family of $G_2$-structures $(\varphi_t, g_t)$ depending on a real variable $t \in (0, \epsilon)$. These $G_2$-structures are not torsion-free, but when $t$ is small, they have small torsion. As $t \to 0$, the $G_2$-structure $(\varphi_t, g_t)$ converges to the singular $G_2$-structure $\pi^*(\varphi_0, g_0)$.

Step 4. We prove using analysis that for all sufficiently small $t$, the $G_2$-structure $(\varphi_t, g_t)$ on $M$, with small torsion, can be deformed to a $G_2$-structure $(\tilde{\varphi}_t, \tilde{g}_t)$, with zero torsion. Finally, we show that $\tilde{g}_t$ is a metric with holonomy $G_2$ on the compact 7-manifold $M$.

We will now explain the steps in greater detail.

Step 1

Here is an example of a suitable group $\Gamma$. Let $(x_1, \ldots, x_7)$ be coordinates on $T^7 = \mathbb{R}^7/\mathbb{Z}^7$, where $x_i \in \mathbb{R}/\mathbb{Z}$. Let $(\varphi_0, g_0)$ be the flat $G_2$-structure on $T^7$ defined by (2). Let $\alpha$, $\beta$ and $\gamma$ be the involutions of $T^7$ defined by

$$\alpha((x_1, \ldots, x_7)) = (-x_1, -x_2, -x_3, -x_4, x_5, x_6, x_7),$$

$$\beta((x_1, \ldots, x_7)) = (-x_1, \frac{1}{2} - x_2, x_3, x_4, -x_5, -x_6, x_7),$$

$$\gamma((x_1, \ldots, x_7)) = (\frac{1}{2} - x_1, x_2, \frac{1}{2} - x_3, x_4, -x_5, x_6, -x_7).$$

By inspection, $\alpha$, $\beta$ and $\gamma$ preserve $(\varphi_0, g_0)$, because of the careful choice of exactly which signs to change. Also, $\alpha^2 = \beta^2 = \gamma^2 = 1$, and $\alpha$, $\beta$ and $\gamma$ commute. Thus they generate a group $\Gamma = \langle \alpha, \beta, \gamma \rangle \cong \mathbb{Z}_2^3$ of isometries of $T^7$ preserving the flat $G_2$-structure $(\varphi_0, g_0)$.

**Lemma 2.1** The elements $\beta \gamma$, $\gamma \alpha$, $\alpha \beta$ and $\alpha \beta \gamma$ of $\Gamma$ have no fixed points on $T^7$.

The fixed points of $\alpha$, $\beta$, $\gamma$ are each 16 copies of $T^3$. The singular set $S$ of $T^7/\Gamma$ is a disjoint union of 12 copies of $T^3$, 4 copies from each of $\alpha$, $\beta$, $\gamma$. Each component of $S$ is a singularity modelled on that of $T^3 \times \mathbb{C}^2/\{\pm 1\}$.  


Thus the singular set splits into a disjoint union of connected components, and each component is very simple. This is helpful because we can desingularize each connected component independently, and simple singularities are easier to resolve.

**Step 2**

Our goal is to resolve the singular set $S$ of $T^7/\Gamma$ to get a compact 7-manifold $M$ with holonomy $G_2$. How can we do this? In general we cannot, because we have no idea of how to resolve general orbifold singularities with holonomy $G_2$. However, suppose we can arrange that every connected component of $S$ is locally isomorphic to either

(a) $T^3 \times \mathbb{C}^2/G$, for $G$ a finite subgroup of $SU(2)$, or  
(b) $\mathcal{S}^1 \times \mathbb{C}^3/G$, for $G$ a finite subgroup of $SU(3)$ acting freely on $\mathbb{C}^3 \setminus 0$.

In this case we can use complex algebraic geometry to find a natural resolution $X$ of $\mathbb{C}^2/G$ or $Y$ of $\mathbb{C}^3/G$, and then $T^3 \times X$ or $\mathcal{S}^1 \times Y$ gives a local model for how to resolve the corresponding component of $S$ in $T^7/\Gamma$.

In case (a), $X$ must have a Kähler metric $h$ with holonomy $SU(2)$ that is asymptotic to the flat Euclidean metric on $\mathbb{C}^2/G$. Such metrics are called Asymptotically Locally Euclidean (ALE). They have been classified by Kronheimer [10, 11], and they exist for every finite subgroup $G \subset SU(2)$. The point is that if $X$ has holonomy $SU(2)$, then the product 7-manifold $T^3 \times X$ has holonomy $\{(1)\} \times SU(2)$. But $\{(1)\} \times SU(2)$ is a subgroup of $G_2$, and so $T^3 \times X$ has a torsion-free $G_2$-structure by Proposition 1.3. Hence, $T^3 \times X$ gives a local model for how to resolve the singularity $T^3 \times \mathbb{C}^2/G$ with holonomy $G_2$.

In case (b), $Y$ is a crepant resolution of $\mathbb{C}^3/G$, and carries an ALE Kähler metric $h$ with holonomy $SU(3)$. Such resolutions and metrics exist for all finite $G \subset SU(3)$, by work of Roan [12] and the author [8]. Since $\{(1)\} \times SU(3) \subset G_2$, if $(Y, h)$ has holonomy $SU(3)$ then $\mathcal{S}^1 \times Y$ has a torsion-free $G_2$-structure, and provides a local model for how to resolve the singularity $\mathcal{S}^1 \times \mathbb{C}^3/G$ with holonomy $G_2$.

Suppose that all the singularities of $T^7/\Gamma$ are of type (a) or (b). Then we can construct a compact, nonsingular 7-manifold $M$ by resolving each singularity $T^3 \times \mathbb{C}^2/G$ using $T^3 \times X$, and resolving each singularity $\mathcal{S}^1 \times \mathbb{C}^3/G$ using $\mathcal{S}^1 \times Y$, as above. In the example this means gluing 12 copies of $T^3 \times X$ into $T^7/\Gamma$, where $X$ is the blow-up of $\mathbb{C}^2/\{\pm 1\}$ at its singular point.

**Step 3**

For each resolution $X$ of $\mathbb{C}^2/G$ in case (a), and $Y$ of $\mathbb{C}^3/G$ in case (b), we can find a 1-parameter family $\{h_t : t > 0\}$ of metrics with the properties

(a) $h_t$ is a Kähler metric on $X$ with $\text{Hol}(h_t) = SU(2)$. Its injectivity radius satisfies $\delta(h_t) = O(t)$, its Riemann curvature satisfies $\|R(h_t)\|_{C^0} = O(t^{-2})$, and $h_t = h + O(t^4 r^{-4})$ for large $r$, where $h$ is the Euclidean metric on $\mathbb{C}^2/G$, and $r$ the distance from the origin.
\(h_t\) is \(\text{Kähler}\) on \(Y\) with \(\text{Hol}(h_t) = SU(3)\), satisfying \(\delta(h_t) = O(t)\), \(\|R(h_t)\|_{C^0} = O(t^{-2})\), and \(h_t = h + O(t^6r^{-6})\) for large \(r\).

In fact we can choose \(h_t\) to be isometric to \(t^2h_1\), and the properties above are easy to prove.

Suppose one of the components of the singular set \(S\) of \(T^7/\Gamma\) is locally modelled on \(T^3 \times \mathbb{C}^2/G\). Then \(T^3\) has a natural flat metric \(h_{T^3}\). Let \(X\) be the resolution of \(\mathbb{C}^2/G\) and let \(\{h_t : t > 0\}\) satisfy property (a). Then \(\tilde{g}_t = h_{T^3} + h_t\) is a metric on \(T^3 \times X\) with holonomy \(\{1\} \times SU(2)\), which is contained in \(G_2\). Thus there is an associated torsion-free \(G_2\)-structure \((\tilde{\varphi}_t, \tilde{g}_t)\) on \(T^3 \times X\). Similarly, if a component of \(S\) is modelled on \(S^1 \times \mathbb{C}^4/G\), we get a family of torsion-free \(G_2\)-structures \((\tilde{\varphi}_t, \tilde{g}_t)\) on \(S^1 \times Y\).

The idea is to make a \(G_2\)-structure \((\varphi_t, g_t)\) on \(M\) by gluing together the torsion-free \(G_2\)-structures \((\tilde{\varphi}_t, \tilde{g}_t)\) on the patches \(T^3 \times X\) and \(S^1 \times Y\), and \((\varphi_0, g_0)\) on \(T^7/\Gamma\). The gluing is done using a partition of unity. Naturally, the first derivative of the partition of unity introduces ‘errors’, so that \((\varphi_t, g_t)\) is not torsion-free.

The size of the torsion \(\nabla \tilde{\varphi}_t\) depends on the difference \(\tilde{\varphi}_t - \varphi_0\) in the region where the partition of unity changes. On the patches \(T^3 \times X\), since \(h_t - h = O(t^4r^{-4})\) and the partition of unity has nonzero derivative when \(r = O(1)\), we find that \(\nabla \varphi_t = O(t^4)\). Similarly \(\nabla \varphi_t = O(t^6)\) on the patches \(S^1 \times Y\), and so \(\nabla \varphi_t = O(t^4)\) on \(M\).

For small \(t\), the dominant contributions to the injectivity radius \(\delta(g_t)\) and Riemann curvature \(R(g_t)\) are made by those of the metrics \(h_t\) on \(X\) and \(Y\), so we expect \(\delta(g_t) = O(t)\) and \(\|R(g_t)\|_{C^0} = O(t^{-2})\) by properties (a) and (b) above. In this way we prove the following result, which gives the estimates on \((\varphi_t, g_t)\) that we need.

**Theorem A** On the compact 7-manifold \(M\) described above, and on many other 7-manifolds constructed in a similar fashion, one can write down the following data explicitly in coordinates:

- Positive constants \(A_1, A_2, A_3\) and \(\epsilon\),
- A \(G_2\)-structure \((\varphi_t, g_t)\) on \(M\) with \(d\varphi_t = 0\) for each \(t \in (0, \epsilon)\), and
- A 3-form \(\psi_t\) on \(M\) with \(d^*\psi_t = d^*\varphi_t\) for each \(t \in (0, \epsilon)\).

These satisfy three conditions:

(i) \(\|\psi_t\|_{L^2} \leq A_1 t^4\) and \(\|d^*\psi_t\|_{L^{14}} \leq A_1 t^4\),

(ii) the injectivity radius \(\delta(g_t)\) satisfies \(\delta(g_t) \geq A_2 t\),

(iii) the Riemann curvature \(R(g_t)\) of \(g_t\) satisfies \(\|R(g_t)\|_{C^0} \leq A_3 t^{-2}\).

Here the operator \(d^*\) and the norms \(\|\cdot\|_{L^2}, \|\cdot\|_{L^{14}}\) and \(\|\cdot\|_{C^0}\) depend on \(g_t\).

Here one should regard \(\psi_t\) as a first integral of the torsion \(\nabla \varphi_t\) of \((\varphi_t, g_t)\). Thus the norms \(\|\psi_t\|_{L^2} \leq A_1 t^4\) and \(\|d^*\psi_t\|_{L^{14}} \leq A_1 t^4\) are measures of \(\nabla \varphi_t\). So parts (i)-(iii) say that the torsion \(\nabla \varphi_t\) must be small compared to the injectivity radius and Riemann curvature of \((M, g_t)\).
STEP 4

We prove the following analysis result.

**Theorem B** In the situation of Theorem A there are constants $\kappa, K > 0$ depending only on $A_1, A_2, A_3$ and $\epsilon$, such that for each $t \in (0, \kappa]$ there exists a smooth, torsion-free $G_2$-structure $(\hat{\varphi}_1, \hat{g}_1)$ on $M$ with $\|\hat{\varphi}_1 - \varphi_i\|_{C^0} \leq Kt^{1/2}$.

Basically, this result says that if $(\varphi, g)$ is a $G_2$-structure on $M$, and the torsion $\nabla \varphi$ is sufficiently small, then we can deform to a nearby $G_2$-structure $(\hat{\varphi}, \hat{g})$ that is torsion-free. Here is a sketch of the proof of Theorem B, ignoring several technical points. The proof is that given in [8], which is an improved version of the proof in [5]. For simplicity we omit the subscripts $t$.

We have a 3-form $\varphi$ with $d\varphi = 0$ and $d^* \varphi = d^* \psi$ for small $\psi$, and we wish to construct a nearby 3-form $\hat{\varphi}$ with $d\hat{\varphi} = 0$ and $d^* \hat{\varphi} = 0$. Set $\hat{\varphi} = \varphi + d\eta$, where $\eta$ is a small 2-form. Then $\eta$ must satisfy a nonlinear p.d.e., which we write as

$$d^* d\eta = -d^* \psi + d^* F(d\eta),$$

where $F$ is nonlinear, satisfying $F(d\eta) = O(|d\eta|^2)$.

We solve (6) by iteration, introducing a sequence $\{\eta_j\}_{j=0}^\infty$ with $\eta_0 = 0$, satisfying the inductive equations

$$d^* d\eta_{j+1} = -d^* \psi + d^* F(d\eta_j), \quad d^* \eta_{j+1} = 0. \tag{7}$$

If such a sequence exists and converges to $\eta$, then taking the limit in (7) shows that $\eta$ satisfies (6), giving us the solution we want.

The key to proving this is an inductive estimate on the sequence $\{\eta_j\}_{j=0}^\infty$. The inductive estimate we use has three ingredients, the equations

1. $$\|d\eta_{j+1}\|_{L^2} \leq \|\psi\|_{L^2} + C_1 \|d\eta_j\|_{L^2} \|d\eta_j\|_{C^0}, \tag{8}$$
2. $$\|\nabla d\eta_{j+1}\|_{L^{14}} \leq C_2 \left( \|d^* \psi\|_{L^{14}} + \|\nabla d\eta_j\|_{L^{14}} \|d\eta_j\|_{C^0} + t^{-4} \|d\eta_{j+1}\|_{L^2} \right), \tag{9}$$
3. $$\|d\eta_j\|_{C^0} \leq C_3 (t^{1/2} \|\nabla d\eta_j\|_{L^{14}} + t^{-7/2} \|d\eta_j\|_{L^2}). \tag{10}$$

Here $C_1, C_2, C_3$ are positive constants independent of $t$. Equation (8) is obtained from (7) by taking the $L^2$-inner product with $\eta_{j+1}$ and integrating by parts. Using the fact that $d^* \varphi = d^* \psi$ and $\psi$ is $O(t^4)$, we get a powerful a priori estimate of the $L^2$-norm of $d\eta_{j+1}$.

Equation (9) is derived from an elliptic regularity estimate for the operator $d + d^*$ acting on 3-forms on $M$. Equation (10) follows from the Sobolev embedding theorem, since $L^{14}_1(M)$ embeds in $C^0(M)$. Both (9) and (10) are proved on small balls of radius $O(t)$ in $M$, using parts (ii) and (iii) of Theorem A, and this is where the powers of $t$ come from.

Using (8)-(10) and part (i) of Theorem A we show that if

$$\|d\eta_j\|_{L^2} \leq C_4 t^4, \quad \|\nabla d\eta_j\|_{L^{14}} \leq C_5, \quad \|d\eta_j\|_{C^0} \leq K t^{1/2}, \tag{11}$$

where $C_4, C_5$ and $K$ are positive constants depending on $C_1, C_2, C_3$ and $A_1$, and if $t$ is sufficiently small, then the same inequalities (11) apply to $d\eta_{j+1}$. Since $\eta_0 = 0$,
by induction (11) applies for all $j$ and the sequence $\{d\eta_j\}_{j=0}^\infty$ is bounded in the Banach space $L^1(A^3T^*M)$. One can then use standard techniques in analysis to prove that this sequence converges to a smooth limit $d\eta$. This concludes the sketch proof of Theorem B.

From Theorems A and B we see that the compact 7-manifold $M$ constructed in Step 2 admits torsion-free $G_2$-structures $(\tilde{\varphi}, \tilde{g})$. Proposition 1.4 then shows that $\text{Hol}(\tilde{g}) = G_2$ if and only if $\pi_1(M)$ is finite. In the example above $M$ is simply-connected, and so $\pi_1(M) = \{1\}$ and $M$ has metrics with holonomy $G_2$, as we want.

By considering different groups $\Gamma$ acting on $T^7$, and also by finding topologically distinct resolutions $M_1, \ldots, M_k$ of the same orbifold $T^7/\Gamma$, we can construct many compact Riemannian 7-manifolds with holonomy $G_2$. Here is a graph of the Betti numbers $b^2(M)$ and $b^3(M)$ of the 68 examples found in [5, 6]. More examples will be given in [8].

On this graph the symbol ‘•’ denotes the Betti numbers of a simply-connected 7-manifold, ‘◦’ denotes a non-simply-connected manifold, and ‘+’ denotes both a simply-connected and a non-simply-connected manifold.

So far we have discussed only the holonomy group $G_2$. There is a very similar construction for compact manifolds with holonomy $\text{Spin}(7)$, described in [7] and [8]. Here are some of the similarities and differences in the two cases. The holonomy group $\text{Spin}(7)$ is a subgroup of $SO(8)$, a compact 21-dimensional Lie group isomorphic to the double cover of $SO(7)$. It is the subgroup of $GL(8, \mathbb{R})$ preserving a certain 4-form $\Omega_0$ on $\mathbb{R}^8$, and also preserves the Euclidean metric $g_0$ on $\mathbb{R}^8$.

Thus a $\text{Spin}(7)$-structure on an 8-manifold $M$ is equivalent to a pair $(\Omega, g)$, where $\Omega$ is a 4-form and $g$ a Riemann metric that are pointwise isomorphic to $\Omega_0$ and $g_0$. Riemannian manifolds with holonomy $\text{Spin}(7)$ are Ricci-flat. Compact manifolds $M$ with holonomy $\text{Spin}(7)$ are simply-connected spin manifolds, and
computing the index of the Dirac operator shows that their Betti numbers must satisfy $b^3(M) + b^4(M) = b^2(M) + b^4(M) + 25$.

We can construct compact 8-manifolds with holonomy $Spin(7)$ by resolving the singularities of orbifolds $T^8/\Gamma$. The construction is more difficult than the $G_2$ case in two ways. Firstly, it seems to be more difficult to find suitable orbifolds $T^8/\Gamma$, and it is necessary to consider more complicated kinds of orbifold singularities. Secondly, the analysis is more difficult, and one has to try harder to make the sequence converge. In [7] we find at least 95 topologically distinct compact 8-manifolds with holonomy $Spin(7)$, realizing 29 distinct sets of Betti numbers, and [8] will give more examples.

Note that compact manifolds with holonomy $G_2$ and $Spin(7)$ are examples of compact Ricci-flat Riemannian manifolds. In fact, compact manifolds with holonomy $G_2$ are the only known source of odd-dimensional examples of compact, simply-connected Ricci-flat Riemannian manifolds.

3 Directions for future research

Here are four areas in which I hope to see interesting developments soon.

- **Other constructions of compact manifolds with exceptional holonomy.** The author has extended the constructions of [5]-[7] to include resolutions of more general quotient singularities, in particular non-isolated quotient singularities $\mathbb{C}^m/G$ for $G$ a finite subgroup of $SU(m)$ and $m = 3$ or 4, and the results will be published in [8]. Another promising possibility is to try to replace the orbifold $T^7/\Gamma$ by $(S^1 \times W)/\Gamma$, where $W$ is a Calabi-Yau 3-fold.

- **Harvey and Lawson’s theory of calibrated geometry** [4] singles out three classes of special submanifolds in manifolds of exceptional holonomy: associative 3-folds and coassociative 4-folds in $G_2$-manifolds, and Cayley 4-folds in $Spin(7)$-manifolds. They are minimal submanifolds, and have good properties under deformation. Compact examples can be constructed as the fixed point sets of isometries, as in [6].

   It would be interesting to study families of compact manifolds of these types, to understand the way singularities develop in such families, and whether a compact $G_2$ or $Spin(7)$-manifold can be fibred by coassociative or Cayley 4-manifolds, with some singular fibres.

- **Gauge theory on compact $Spin(7)$-manifolds.** Let $M$ be a compact 8-manifold with holonomy $Spin(7)$, let $E$ be a vector bundle or principal bundle over $M$, and let $A$ be a connection on $E$. Then the curvature $F_A$ of $A$ is a 2-form with values in $\text{ad}(E)$. Now the $Spin(7)$-structure induces a splitting $\Lambda^2 T^* M = \Lambda_7^2 \oplus \Lambda_{21}^2$, where $\Lambda_7^2, \Lambda_{21}^2$ are vector bundles over $M$ with fibre $\mathbb{R}^7$, $\mathbb{R}^{21}$ respectively. We call $A$ a $Spin(7)$-instanton if the component of $F_A$ in $\text{ad}(E) \otimes \Lambda_7^2$ is zero.

   It turns out that $Spin(7)$-instantons have many properties in common with instantons in 4 dimensions, that are studied in Donaldson theory. Christo-
pher Lewis and the author [9] have proved an existence theorem for Spin(7)-
stantons with gauge group SU(2) on certain compact 8-manifolds with
holonomy Spin(7). In 4 dimensions a sequence of instantons can ‘bubble’
at a finite number of points. In 8 dimensions we expect ‘bubbling’ to occur
instead around a compact Cayley 4-manifold, and we construct families of
instantons in which this happens.

• Connections with String Theory. String Theory is a branch of high-energy
theoretical physics that aims to unify quantum theory and gravity by mod-
elling particles as 1-dimensional objects called strings. One of its features is
that it prescribes the dimension of space-time. This depends on the details of
the theory, but the most popular model, supersymmetric string theory, gives
dimension 10. To explain the discrepancy between this and the 4 space-time
dimensions that we observe, it is supposed that the universe looks locally
like \( \mathbb{R}^4 \times M^6 \), where \( M^6 \) is a compact 6-manifold with very small radius, of
order \( 10^{-33} \) cm.

In supersymmetric string theory, \( M \) must be a Calabi-Yau 3-fold. So string
theorists are interested in Calabi-Yau 3-folds, and have contributed many
ideas to the subject, including that of Mirror Symmetry. However, if in-
stead we consider \( \mathbb{R}^3 \times M^7 \), corresponding to an observable universe with 3
space-time dimensions, then by work of Vafa and Shatashvili \( M^7 \) must be a
compact 7-manifold with holonomy \( G_2 \). Similarly, if we consider \( \mathbb{R}^2 \times M^8 \),
so that the observable universe has 2 space-time dimensions, then \( M^8 \) is a
compact 8-manifold with holonomy \( \text{Spin}(7) \).

Recently, string theorists have begun to seriously consider the possibility that
the universe may have 11 dimensions (‘M theory’) or even 12 dimensions (‘F
theory’). To reduce to 4 observable space-time dimensions in these theories
will require a manifold of dimension 7 or 8, and it seems likely that compact
manifolds with exceptional holonomy will play a rôle in this.

References

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