Dynamics, Topology, and Holomorphic Curves

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Abstract. In this paper we describe the intimate interplay between certain classes of dynamical systems and a holomorphic curve theory. There are many aspects touching areas like Gromov-Witten invariants, quantum cohomology, symplectic homology, Seiberg-Witten invariants, Hamiltonian dynamics and more. Emphasized is this interplay in real dimension three. In this case the methods give a tool to construct global surfaces of section and generalizations thereof for the large class of Reeb vector fields. This class of vector fields, includes, in particular, all geodesic flows on surfaces.

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1 Periodic orbits of dynamical systems

Symplectic and contact geometry as well as Hamiltonian dynamics experienced in the last decade a tremendous growth. In order to cover some aspects in a certain depth one faces the serious dilemma of making a selection. Rather than touching many areas, it seems more appropriate to focus only on a few aspects. The choice made here was to describe the subtle relationship between Hamiltonian dynamics, topology and a theory of holomorphic curves. So many aspects are only briefly mentioned or even ignored. However, they are being dealt with in other papers contained in the proceedings of the ICM Berlin. In particular the contributions by S. Donaldson, Y. Eliashberg, K. Kuperberg, D. McDuff, J. Moser, L. Polterovich, Y. Ruan and C. Taubes.

The aim of this paper is to explain some of the recent progress at the interface of Hamiltonian dynamics and symplectic geometry. In order to appreciate the special features of (certain) Hamiltonian dynamics versus general dynamics we begin with the following classical problem.

In 1950 Seifert, [79], raised the question if a given non-singular vector field X on the three-sphere admits a periodic orbit:

$$\dot{x} = X(x) \text{ and } x(0) = x(T),\ T > 0.$$

As it turned out this is a subtle problem. In higher dimensions Wilson, [87], provided in 1966 examples of non-singular vector fields on S^{2n-1}, $n \geq 3$, without periodic orbits. However, dimension three poses more difficulties due to lack of room in order to make some of the higher dimensional ideas work. After all, destroying periodic orbits, which are 1-dimensional sets, should be easier in higher dimensions.

In 1974 Schweizer, [77], showed that there exist non-singular C^1-vector fields on S^3 without any periodic orbits. The regularity of the counterexample was strengthened to C^2 in [36]. In 1994 the question was finally settled by K. Kuperberg, [59], who constructed a real analytic counterexample.

Theorem 1.1 (K. Kuperberg) *There exists a nowhere vanishing real analytic vector field on S^3 without any periodic orbit.*

So, asking for periodic orbits, given an arbitrary smooth vector field on S^3 (and as the method shows on any three-manifold) is not a good question if we only know little about the dynamical system. On the other hand, at the end of the seventies most notably by Rabinowitz, [74, 75], and Weinstein, [85], there were some positive results concerning special vector fields coming from Hamiltonian systems. Rabinowitz's somewhat more general result is the following:

Theorem 1.2 (Rabinowitz) *A regular energy surface of an autonomous Hamiltonian system in $\mathbf{R}^{2n}$, which bounds a star-shaped domain, carries a periodic orbit.*

Weinstein proved a slightly weaker result assuming that the energy surface bounds a convex domain.

Figure 1: *A starshaped energy surface is diffeomorphic to a sphere centered at some point via radial transformation.*

We note that from a symplectic purist's point of view the results are not satisfactory, since the assumptions are not invariant under symplectic (or canonical) transformations.

Abstractly speaking we have here an existence result for certain non-singular vector fields on spheres S^{2n-1}. What is interesting now, of course, is the cut-off line between "Guaranteed Existence" and "Possible Non-Existence".

Based on the above mentioned results by Rabinowitz and his own contribution, Weinstein made in 1978 a conjecture, [86], which together with the earlier Arnold conjectures, [2], in symplectic fixed point theory had a tremendous impact.

Rabinowitz's result were extremely important, in particular psychologically, since the degenerate and indefinite classical Hamiltonian variational principle was used for the first time to study existence problem of periodic orbits in Hamiltonian dynamics. One should keep in mind that this variational principle was thought to be only formal and completely useless for existence questions.

There were a certain number of difficulties to overcome. First of all one had to find a suitable functional analytic set-up, secondly one had to deal with the problem that apriori the Morse indices of the critical points of the functional had infinite Morse-index and co-index, so that (Palais-Smale type) Morse-theory in infinite-dimensional spaces would indicate that there is no relationship between the critical points and the topology of the underlying space, which in our case is the free loop space of the underlying symplectic manifold. Shortly afterwards Conley and Zehnder, [15], showed how the action principle by means of the Conley-index theory could be used to do symplectic fixed point theory, by proving a symplectic fixed point theorem for tori. Extensions of the methods for more general manifolds were however obstructed by immense technical difficulties. In 1985, Gromov, [34], introduced PDE-methods to symplectic geometry (the theory of pseudoholomorphic curves), "ignoring" however the underlying variational structure. (The word "ignoring" might be somewhat too strong here. The variational structure enters in the theory in the disguise of area bounds, which are of course extremely (in fact intrinsically) important in Gromov's theory.)

Then in 1987, Floer, [25], brought together the Conley-Zehnder variational point of view and Gromov's PDE methods and constructed his famous (symplectic) Floer homology theory. After that there were still some serious obstacles to overcome. For example, that the symplectic fixed point problem is not variational in general, but rather comparable with doing Morse-theory for a closed 1-form. (This calls for a Novikov-type Floer-theory, which was carried out in [43].) Besides that, the notorious difficulty of understanding holomorphic spheres in symplectic manifolds and in particular multiple covered spheres hindered progress for quite a while. Recently these difficulties were overcome, see in particular [30, 64, 63].

After this historical excursion let us state the Weinstein conjecture.

Conjecture 1.3 (Weinstein) *Let M^{2n-1} be a $(2n-1)$-dimensional closed manifold and X a non-singular smooth vector field. Assume there exists a 1-form λ having the following properties:*

$$\lambda \wedge d\lambda^{n-1} \text{ is a volume form,}$$
$$d\lambda(X, \cdot) = 0,$$
$$\lambda(X) > 0.$$

Then X has a periodic orbit.

We call a 1-form λ a contact form if $\lambda \wedge d\lambda^{n-1}$ is a volume form. We observe that a contact form defines a non-singular vector field X by

$$i_X d\lambda = 0 \text{ and } i_X\lambda = 1. \tag{1}$$

This uniquely determined vector field X is called the Reeb vector field associated with the contact form λ. Clearly, if $f : M \to (0, \infty)$ is a smooth function, then the vector field X admits a periodic orbit if and only if fX admits a periodic orbit.

Therefore there is no loss of generality assuming in the Weinstein conjecture that X is a Reeb vector field. Note that λ defines a hyperplane distribution $\xi \to M$ by

$$\xi = \text{kern}(\lambda).$$

This plane field distribution is completely non-integrable. It is called a contact structure. We refer the reader to Arnold's book, [3], appendix 4, for more basic information about contact structures.

For our purposes we note here, that given a contact form on a three-dimensional manifold, there can always be introduced local coordinates (x, y, z) in which the contact form λ is given by $\lambda = dz + xdy$.

Figure 2: *The local model for a contact structure in dimension three*

At first glance the hypothesis in the Weinstein conjecture seems mysterious. However, some work reveals that it is has a geometrically compelling meaning. Namely M may be viewed as an element of a smooth 1-parameter family of mutually different Hamiltonian energy surfaces M_δ, $\delta \in [-\delta_0, \delta_0]$ (with $M_0 = M$) in $[-\delta_0, \delta_0] \times M$ equipped with a suitable symplectic structure, so that flows on two different energy surfaces are conformally symplectically the same. In particular the flows on any two such energy surfaces are conjugated.

So, for example, if one of these energy surfaces contains a periodic orbit, so do the others. One can reformulate the Weinstein conjecture as follows.

CONJECTURE 1.4 (WEINSTEIN) *Assume (W, ω) is a symplectic manifold and $H : W \to \mathbf{R}$ a smooth Hamiltonian, so that $M := H^{-1}(E)$ is a compact regular energy surface (for some energy E). If there exists a 1-form λ on M such that $\lambda(X_H(x)) \neq 0$ for $x \in M$ and $d\lambda = \omega|M$, then there exists a periodic orbit on M.*

Here X_H is the Hamiltonian vector field defined by

$$i_{X_H}\omega = -dH.$$

Both formulations of the Weinstein conjecture are equivalent.

Having this in mind one can appreciate the following result. Consider the symplectic vector space $(\mathbf{R}^{2n}, \omega)$. The symplectic form is defined by

$$\omega = \sum_{j=1}^{n} dq_j \wedge dp_j.$$

Recall, that given an autonomous Hamiltonian $H : \mathbf{R}^{2n} \to \mathbf{R}$ we have the associated Hamiltonian system

$$\dot{z} = X_H(z). \tag{2}$$

Denote by Σ_H the set of all $E \in \text{image}(H)$ such that there exists no periodic solution (z, T) of (2) with $H(z) = E$. Now the following almost existence result holds, which tells us that periodic orbits are a common phenomenon and that there are usually many of them.

Theorem 1.5 *Let $H : \mathbf{R}^{2n} \to \mathbf{R}$ be a smooth Hamiltonian satisfying $H(z) \to \infty$ if $|z| \to \infty$. Then measure$(\Sigma_H) = 0$.*

This theorem was essentially proved by Hofer and Zehnder, [54], where it was shown that the complement of Σ_H is dense. The same approach was then pushed to its limits by Struwe, [81], showing that measure$(\Sigma_H) = 0$.

This almost existence phenomenon can be understood best within the symplectic capacity theory, see [56]. It holds for more general symplectic manifolds. However, not for all manifolds, see [37] for some very interesting phenomena.

Nevertheless one might ask if a regular compact energy surface necessarily carries a periodic orbit. We begin with a positive application. Using Theorem 1.5 we can recover Viterbo's landmark result, namely the proof of the Weinstein conjecture in $\mathbf{R}^{2n}$, [84]:

Corollary 1.6 (Viterbo) *Given a closed, connected hypersurface M in $(\mathbf{R}^{2n}, \omega)$, admitting a contact form λ such that $d\lambda = \omega|M$, any Hamiltonian system having M as a regular energy surface admits a periodic orbit on M.*

The proof, based on Theorem 1.5 is obvious. Foliate the neighborhood of M by conformally symplectic images M_δ, $\delta \in [-1, 1]$ by using the contact hypothesis. Assume that M_{-1} is contained in the bounded component B of $\mathbf{R}^{2n} \setminus M$.

Now define a Hamiltonian H having the property that $H^{-1}(\delta) = M_\delta$ for $\delta \in [-\frac{1}{2}, \frac{1}{2}]$, so that these M_δ are regular energy surfaces. In addition $H(z) \to \infty$ for $|z| \to \infty$.

Figure 3: *The level sets for the constructed Hamiltonian H. The Hamiltonian is constant between a big sphere S and M_1.*

An application of Theorem 1.5 shows that for some $\delta \in [-\frac{1}{2}, \frac{1}{2}]$ there exists a periodic orbit. Since all these hypersurfaces are conformally symplectically equivalent there is also one on $M_0 = M$. So the theorem can be used to prove existence results. But is the theorem optimal?

By results of Ginzburg, [31, 32], and Herman, [38, 39] the following holds.

Theorem 1.7 *For $n \geq 3$ there exists a smooth embedding Φ of $[-1,1] \times S^{2n-1}$ into $(\mathbf{R}^{2n}, \omega)$, such that $M_0 = \Phi(\{0\} \times S^{2n-1})$ does not contain any periodic solution.*

By the almost existence theorem, of course,

$$measure\{\delta \in [-1,1] \mid M_\delta \text{ contains a periodic orbit}\} = 2.$$

So, in some sense, in $\mathbf{R}^{2n}$, for $n \geq 3$, the almost existence result is the best possible. Nevertheless it is still an open question if Theorem 1.7 holds for $n = 2$.

At this point we have almost existence results and non-existence results and an existence result for closed contact type hypersurfaces in $\mathbf{R}^{2n}$.

Are there manifolds for which one can say that every Reeb vector field on them has a periodic orbit?

Theorem 1.8 (Hofer) *Assume that X is a Reeb vector field on a closed three-manifold M. Then X admits a periodic orbit if either M is finitely covered by S^3, or if $\pi_2(M) \neq \{0\}$, or if the underlying contact structure is overtwisted.*

The notion of an overtwisted contact structure is important in three-dimensional contact geometry.

Definition 1.9 *Let λ be a contact structure on the three-manifold M with underlying contact structure $\xi = kern(\lambda)$. The contact structure ξ is said to be overtwisted if there exists an embedded disk $\mathcal{D} \subset M$, such that*

$$T(\partial\mathcal{D}) \subset \xi|(\partial\mathcal{D}) \tag{3}$$

$$T_z\mathcal{D} \not\subset \xi_z \text{ for all } z \in \partial\mathcal{D}. \tag{4}$$

We call a contact structure tight if it is not overtwisted. (Figure 4 gives an example of an overtwisted disk).

It is a fundamental result by Bennequin, [6], that the so-called standard contact structure on S^3

$$\lambda_0 = \frac{1}{2}[q \cdot dp - p \cdot dq]|S^3$$

is tight.[1]

In a deep paper (which stunned many of the experts), [20], Eliashberg classified all overtwisted contact structures for a closed three-manifold M. This classification can be done in purely homotopy theoretic terms.[2] In addition he showed

[1] Here S^3 is viewed as the unit sphere in $\mathbf{C}^2$, where the latter is equipped with the coordinates $z = q + ip$, $q, p \in \mathbf{R}^2$.

[2] There is an “h-principle” in the background.

Figure 4: *An overtwisted contact structure on* $\mathbf{R}^3$.

that up to diffeomorphism there is only one (positive) tight contact structure on S^3 but a countable number of overtwisted contact structures and also classified the contact structures on $\mathbf{R}^3$, see [22, 23]. One should also mention the work by Giroux, most notably [33], which had a great impact on contact geometry.

2 Periodic orbits in Hamiltonian dynamics and rigidity

As the preceding discussion shows, finding periodic orbits is an "ill-posed" problem in general, but well-posed" for a certain class of dynamical systems.

From a dynamical systems point of view periodic orbits allow to study the flow in a neighborhood by means of a return map. In the case of a Hamiltonian system one can expect already very striking phenomena as Figure 5 shows. The fixed point 0 in the middle is surrounded by smooth curves, which are invariant under the return map. These curves where discovered by Moser, [71]. Between these curves there are orbits of elliptic and hyperbolic periodic points. The stable and unstable manifolds starting from these hyperbolic points intersect transversally in so-called homoclinic points. Due to these homoclinic points we have invariant hyperbolic sets on which the iterates of the return map behave like a Bernoulli shift. The dotted lines represent the recently discovered Mather-sets, [66]. The generic existence of the homoclinic orbits was rigorously established by Zehnder, [90].

Particularly interesting are hyperbolic periodic orbits if they come together with a (global) homoclinic orbit. Then, if the stable and unstable manifold intersect transversally, a very rich dynamics unfolds near the union of the periodic and the homoclinic orbit.

Surprisingly, there is an additional dimension to the periodic orbits, which only in the last ten years has become apparent . Namely the importance of periodic orbits in a symplectic rigidity theory. They are the objects which carry important symplectic information. Let us mention two of these constructions. The first is a symplectic capacity introduced by Hofer and Zehnder, [55]. Consider the category $\mathcal{S}^{2n}$ consisting of all of all $2n$-dimensional symplectic manifolds (with or without boundary) as objects and the symplectic embeddings as morphisms.

For every symplectic manifold (W, ω) in $\mathcal{S}^{2n}$ we consider the collection

Figure 5: *The dynamical complexity near a generic elliptic periodic orbit, as seen for the return map of a transversal section.*

$\mathcal{H}(W, \omega)$ of all smooth maps $H : W \to (-\infty, 0]$ with compact support $\text{supp}(H)$ such that:

- $\text{supp}(H) \cap \partial W = \emptyset$.
- There exists a nonempty open set U with $H|U \equiv const. = \inf_{x \in W} H(x)$.
- Every periodic orbit of the Hamiltonian system $\dot{x} = X_H(x)$ with period $T \in (0, 1]$ is constant.

Then define a number $c(W, \omega) \in [0, \infty) \cup \{\infty\}$ by

$$c(W, \omega) := \sup_{H \in \mathcal{H}(W,\omega)} \parallel H \parallel_{C^0} .$$

These numbers are new symplectic invariants called symplectic capacities and are by their very nature 2-dimensional invariants of the symplectic manifold (W, ω). Of course the volume $\text{vol}(W, \omega) = \int_W \omega^n$ is a $2n$-dimensional invariant. The formal properties of c are:

- If $(W, \omega) \to (V, \tau)$ then $c(W, \omega) \leq c(V, \tau)$.
- $c(W, \alpha\omega) = |\alpha| \cdot c(W, \omega)$ for $\alpha \in \mathbf{R} \setminus \{0\}$.
- $c(B^{2n}) = c(Z^{2n}) = \pi$.

Here B^{2n} is the Euclidean unit ball in $\mathbf{R}^{2n}$ and Z^{2n} the unit-cylinder $B^2 \times \mathbf{R}^{2n-2}$, both equipped with the induced symplectic structure. [3]

If (ϕ_t) is a Hamiltonian flow on some symplectic manifold and U is an open subset then not only the volume of $\text{vol}(\phi_t(U))$ is independent of t but also the symplectic capacity $c(\phi_t(U))$.

[3] The definition of a symplectic capacity is motivated by Gromov's celebrated (non-)squeezing theorem, [34, 35]. His theorem leads to a capacity called "Gromov's width".

As it turns out there are many different constructions for symplectic capacities. Some involve the theory of pseudoholomorphic curves, [34], some the least action principle in Hamiltonian dynamics, [18], and there is even one using a symplectic homology theory, [27]. In reference [18] symplectic rigidity phenomena were shown for the first time to be related to periodic orbit problems.

Symplectic homology is a realization of the following idea. Assume that we consider the usual homology theory, but restricted to the category $\mathcal{S}^{2n}$. Since the spaces have an additional symplectic structure and the morphisms are symplectic embeddings it seems plausible that the restricted standard (topological) homology functor is obtained by composing a forgetful functor with some (much more complex) symplectic homology functor. Indeed, along these lines a symplectic homology functor can be constructed depending on three parameters, namely an integer k and a pair of real numbers $a \leq b$. As it turns out the symplectic homology for sufficiently nice symplectic manifolds W with boundary is constructed out of the topology of M and the periodic orbits for the Hamiltonian flow on ∂W. The action of the periodic orbits gives a real filtration (leading to the a, b-dependence) and the Conley-Zehnder indices of the periodic orbits (a substitute for the Morse index, when seeing periodic orbits as critical points of some Morse function on a suitable loop space) lead to the integer grading. For more details the reader is referred to [27], or to [56] for a short overview.

3 Holomorphic curves and the Weinstein conjecture

As it turns out there is a subtle relationship between the dynamics of Reeb vector fields and an holomorphic curve theory. In order to explain this "holomorphic connection" we start with a specific example. View S^{2n-1} as the unit sphere in $\mathbf{C}^n$. We write the coordinates in $\mathbf{C}^n$ as

$$z = (z_1, ..., z_n) = (q_1 + ip_1, ..., q_n + ip_n)$$

with $z_j \in \mathbf{C}$ and $q_j, p_j \in \mathbf{R}$. The standard contact form λ_0 on S^{2n-1} is defined by:

$$\lambda_0 = \frac{1}{2} \sum_{j=1}^{n} [q_j dp_j - p_j dq_j] | S^3.$$

The Reeb vector field is given by $X_0(z) = 2iz$, which generates the Hopf fibration and the contact structure ξ_0 is the bundle of complex $(n-1)$-planes in TS^{2n-1}.

The idea, which is difficult to motivate apriori, is now the following. Introduce on $\mathbf{R} \times S^{2n-1}$ the complex structure $\tilde{J}$ by requiring that the diffeomorphism

$$\Phi : \mathbf{C}^n \setminus \{0\} \to \mathbf{R} \times S^{2n-1}, z \to (\frac{1}{2}\ell n|z|, \frac{z}{|z|})$$

is biholomorphic, i.e.

$$T\Phi \circ i = \tilde{J} \circ T\Phi.$$

Then, one easily verifies that $\tilde{J}$ is given by

$$\tilde{J}(a,u)(h,k) = (-\lambda_0(u)(k), i\pi_0 k + hX_0(u)), \tag{5}$$

where $\pi_0 : TS^{2n-1} \to \xi_0$ is the projection along X_0. Of course under Φ the study of holomorphic curves in $\mathbf{C}^n$, which avoid the origin is equivalent to studying holomorphic curves in $\mathbf{R} \times S^{2n-1}$. In $\mathbf{C}^n$ there is a very nice class of holomorphic curves, namely the affine algebraic curves. In which way are they distinguished from an arbitrary holomorphic curve? Denote by Σ the collection of all smooth maps $\mathbf{R} \to [0,1]$ having non-negative derivative and associate to $\varphi \in \Sigma$ the 2-form on $\mathbf{R} \times S^{2n-1}$ defined by

$$\omega_\varphi = d(\varphi\lambda_0),$$

where $(\varphi\lambda_0)(a,u)(h,k) = \varphi(a)\lambda_0(u)(k)$.

The interesting observation is now, [42]:

Proposition 3.1 *Assume that A is a connected closed subset of $\mathbf{C}^n \setminus \{0\}$. Then the following statements are equivalent:*

1. *The closure of A in $\mathbf{C}^n$ is an irreducible 1-dimensional affine algebraic curve.*
2. *There exists a connected closed Riemann surface (S,j), and a finite subset $\Gamma \subset S$, and a smooth map $\tilde{u} : S \setminus \Gamma \to \mathbf{R} \times S^{2n-1}$ such that*

$$\begin{gathered}\tilde{J} \circ T\tilde{u} = T\tilde{u} \circ j, \qquad (6)\\ 0 < E(\tilde{u}) := sup_{\varphi\in\Sigma} \int_{S\setminus\Gamma} \tilde{u}^*\omega_\varphi < \infty,\\ \tilde{u} \textit{ cannot be smoothly extended over any point in } \Gamma,\\ \Phi(S) = \tilde{u}(S \setminus \Gamma).\end{gathered}$$

Clearly $T\tilde{u} \circ j = \tilde{J} \circ T\tilde{u}$ is a non-linear Cauchy-Riemann type equation. If $\tilde{u}$ is a solution, then necessarily $\tilde{u}^*\omega_\varphi$ is a non-negative integrand, so that the definition of $E(\tilde{u})$ makes sense. The estimate $E(\tilde{u}) > 0$ implies that $\tilde{u} \not\equiv$ *const.*. The finiteness of the energy means analytically that given a solution $\tilde{u}$ of the Cauchy-Riemann equation and an $\mathbf{R}$-invariant metric on $\mathbf{R} \times S^{2n-1}$, the area of the image of $\tilde{u}$ in any set $[c, c+1] \times S^{2n-1}$ is uniformly bounded independent of $c \in \mathbf{R}$. This, of course, corresponds to polynomial growth if we view the corresponding curve in $\mathbf{C}^n$. What is the behavior near the points in Γ (the punctures)?

Near a (non-removable) puncture z_0 the image of a tiny punctured disk around z_0 is approximately a half-cylinder $[R, \infty) \times P$, where P is a Hopf circle. There is a suggestive way to generalize the above situation. Namely, consider a closed manifold M of dimension $2n-1$, equipped with a contact form λ. Make one choice, by taking a complex multiplication $J : \xi \to \xi$, where $\xi = \mathrm{kern}(\lambda)$, so that

$$g_J(u)(k,k') = d\lambda(u)(k, J(u)k')$$

defines fibre-wise a positive inner product for the bundle $\xi \to M$. Then define an $\mathbf{R}$-invariant almost complex structure on $\mathbf{R} \times M$ by

$$\tilde{J}(a,u)(h,k) = (-\lambda(u)(k), J(u)\pi k + hX(u)),$$

where X is the Reeb vector field associated to λ and $\pi : TM \to \xi$ the projection along X. The definition of E generalizes by replacing $\varphi\lambda_0$ by $\varphi\lambda$. So our new equation becomes

$$\begin{aligned} \tilde{u} &: S \setminus \Gamma \to \mathbf{R} \times M \\ T\tilde{u} \circ j &= \tilde{J} \circ T\tilde{u} \\ 0 &< E(\tilde{u}) < \infty. \end{aligned} \tag{7}$$

What is the behavior near a puncture $z_0 \in \Gamma$? There are three mutually exclusive possibilities.

1. Positive puncture: $\lim_{z \to z_0} a(z) = \infty$
2. Negative puncture: $\lim_{z \to z_0} a(z) = -\infty$
3. Removable puncture: $\lim_{z \to z_0} a(z) =: a_0 \in \mathbf{R}$

In the last case the map $\tilde{u}$ can be smoothly extended over z_0. Let us assume that for the following $\tilde{u}$ has been extended over all removable punctures. We note that for a solution of (7) the set Γ cannot be empty. Indeed, otherwise by Stokes' theorem $E(\tilde{u}) = 0$.

The relationship between the solutions of (7) and the periodic orbits of X is contained in

Theorem 3.2 (Hofer) *Let λ be a contact form on the closed $(2n-1)$-dimensional manifold M and J be an admissible complex multiplication for the underlying contact structure ξ. If (7) has a solution, then there exists a periodic orbit for the Reeb vector field with period $T \leq E(\tilde{u})$.*

For generic λ the finite energy surface approximates near a puncture a cylinder over a periodic orbit. Figure 6 on the next page shows a finite energy surface with two positive punctures and one negative puncture.

In order to use that theorem, one needs to develop methods to find holomorphic curves solving (7). Whereas the first existence results were based on adhoc methods it meanwhile became clear for specialists that there is a (Floer-type) homology theory in the background. It has already been christened "Contact Homology", but doesn't yet exist as "hard copy". This theory was envisioned by Eliashberg and the author in 1993 after the paper [40], and some talks about special cases were given at various places, in particular at the IAS/Park City summer institute on symplectic geometry, [24]. To create such a homology theory for general closed contact manifolds, one encounters certain analytical problems in counting holomorphic curves, quite familiar from the Arnold conjectures. However, since the recent solution of the Arnold conjectures overcomes these difficulties one should be able deal with these problems.

By the previous discussion the Weinstein conjecture has been reduced to finding nontrivial holomorphic curves. The following theorem is the first, dealing with the solvability of (7). The method used is an Eliashberg-type disk-filling, [21], based on Gromov's pseudoholomorphic curve theory. These type of methods are familiar in the theory of several complex variables, where they are used to study envelopes of holomorphy, see [5]. The main point here is, however, that it is apriori known that the analysis involved in the disk-filling has to fail.

Figure 6: *A finite energy surface with* 2 *positive punctures and one negative puncture*

THEOREM 3.3 (HOFER) *Assume M is a closed three-manifold and λ a contact form. Let $J : \xi \to \xi$ be an admissible complex multiplication for the underlying contact structure and denote by $\tilde{J}$ the associated almost complex structure on $\mathbf{R} \times M$. If either $M = S^3$, or $\pi_2(M) \neq 0$, or ξ is overtwisted there exists a solution $\tilde{u}$ of (7) with $S = S^2$ and $\Gamma = \{\infty\}$. In other words there exists a finite energy plane.*

We note that Theorem 3.3 implies Theorem 1.8. The proof is based on a careful analysis of certain nonlinear boundary value problems involving a non-linear Cauchy-Riemann type operator on the disk. One knows for topological reasons that there cannot be apriori estimates and studies carefully how the estimates fail. A bubbling-off analysis making extensive use of the $\mathbf{R}$-invariance of $\tilde{J}$ then allows via reparametrizations to construct these solutions. We refer the reader to the upcoming book [1] for a very detailed description of the methods, and to [40, 41, 42] for the original proof and some discussion of the underlying ideas.

In dimension three we can sometimes say more about the nature of the periodic orbits to which the holomorphic curves are asymptotic. For example, for every Reeb vector field on S^3 there exists an unknotted periodic orbit, see [52].

Many interesting and surprising examples illustrating how bad arbitrary Hamiltonian flows can behave in contrast to Reeb flows can be found in [11].

4 Global systems of surfaces of section

On might wonder, if one can say more about the dynamics. Here we restrict ourselves to the three-dimensional cases for the sole reason that the methods cannot be employed in higher dimensions.

Assume we are given a closed three-manifold M and a nowhere vanishing vector field X. We would like to understand the dynamics. A successful idea going back to Poincaré and Birkhoff, [7], is to find a global surface of section, reducing the understanding of the dynamics to the problem of understanding a self-map of a surface. Of course there are topological and dynamical obstructions in finding such a surface.

Definition 4.1 *A global surface of section for* (M, X) *is a compact surface (perhaps with boundary)* $\Sigma \subset M$*, such that* $\partial\Sigma$ *consists of periodic orbits and* $\dot{\Sigma} = \Sigma \setminus \partial\Sigma$ *is transversal to* X*, so that in addition every orbit other than those in* $\partial\Sigma$ *hit* $\dot{\Sigma}$ *in forward and backward time.*

The surface of section allows to define a return map $\psi : \dot{\Sigma} \to \dot{\Sigma}$. Then the dynamics is encoded in ψ. Of course, having in mind how complicated flows are, one really doesn't expect the existence of such a surface of section. For example, any surface of section for (S^3, X) must necessarily have a boundary. Indeed, if there is no boundary component, S^3 would necessarily fiber over S^1, which by the exact homotopy sequence for a fibration would imply that $\pi_1(S^3) \neq \{1\}$. On the other hand if there is a boundary component there has to be a periodic orbit, which however need not to exist by Kuperberg's result. Also, in the volume-preserving case it is doubtful if something sensible can be said. However, as it turns out, for Reeb flows on three-dimensional manifolds, a whole theory of surfaces of section almost intrinsically exists. This theory, which will be discussed now, should be possible for every (or at least many) three-manifold. However, details have only been carried out so far for S^3.

Let us begin with S^3 equipped with the standard structure λ_0. Again we let ourselves be inspired by the model problem. Denote by

$$\Phi : \mathbf{C}^2 \setminus \{0\} \to \mathbf{R} \times S^3$$

the diffeomorphism

$$z \to (\frac{1}{2}\ell n|z|, \frac{z}{|z|})$$

previously defined.

Consider the sets $\Phi(\mathbf{C} \times \{c\})$, where $c \in \mathbf{C} \setminus \{0\}$, and $\Phi((\mathbf{C} \setminus \{0\}) \times \{0\})$. The union of these sets is a smooth foliation $\mathcal{F}$ of $\mathbf{R} \times S^3$ consisting of finite energy surfaces. Observe that we have a natural $\mathbf{R}$-action:

$$\mathbf{R} \times \mathcal{F} \to \mathcal{F}, (a, F) \to a + F,$$

Figure 7: *The collection of projected surfaces establishes an open book decomposition of* S^3

where

$$a + F := \{(a + b, m) \mid (b, m) \in F\}.$$

There is one fixed point F_0 of this action corresponding to a cylinder over the periodic orbit $P = S^1 \times \{0\}$:

$$F_0 = \mathbf{R} \times P.$$

If the surfaces are projected into S^3 the fixed point F_0 projects onto the Hopf circle P and all other surfaces onto open disks bounded by P. The collection of projected surfaces in fact establishes an open book decomposition of S^3 with disk-like pages, see Figure 7.

What happens if we modify the contact form, but keep the contact structure, i.e. replace λ_0 by $\lambda = f\lambda_0$?

In order to study this question it is useful to make the following definition.

Definition 4.2 *Let M be a closed three-manifold, λ a contact form on M and J a complex multiplication for the associated contact structure. A finite energy foliation $\mathcal{F}$ for (M, λ, J) is a 2-dimensional smooth foliation for $\mathbf{R} \times M$ such that the following holds:*

- *There exists a universal constant $C > 0$ such that for every leaf $F \in \mathcal{F}$ there exists an embedded finite energy curve $(S, \Gamma, \tilde{u})$ for (M, λ, J) satisfying*

$$F = \tilde{u}(S \setminus \Gamma)$$

  and $E(\tilde{u}) \leq C$. All punctures Γ are assumed to be non-removable.

- *For every $a \in \mathbf{R}$ and $F \in \mathcal{F}$ also $a + F$ belongs to $\mathcal{F}$. In particular either $F = F_a$ or $F \cap (a + F) = \emptyset$.*

Let us call a contact form λ non-degenerate if all the periodic orbits (x, T) for X_λ are non-degenerate in the following sense. Denote by η_t the flow associated to

X and observe that it preserves λ, so that the tangent map $T\eta_t(x(0))$ induces a map

$$L_{(x,t)} := T\eta_t(x(0))|\xi_{x(0)} : \xi_{x(0)} \to \xi_{x(T)}.$$

For every period $T > 0$ we therefore obtain a self map of $x_{\xi(0)}$, which is symplectic with respect to the structure $d\lambda(x(0))$. We say (x, T) is non-degenerate if 1 does not belong to the spectrum of $L_{(x,T)}$.

We assume that we are given a closed three-manifold M and a contact form λ with associated Reeb vector field X and contact structure ξ. Assuming that the contact form λ is non-degenerate is a generic condition. Indeed, the following holds.

Proposition 4.3 *Fix a contact form τ on the closed three manifold M. Consider the subset $\Xi_1 \subset C^\infty(M, (0, \infty))$ consisting of all those f such that $\lambda = f\tau$ is non-degenerate. Let Ξ_2 consist of all those $f \in \Xi_1$ such in addition the stable and unstable manifold of hyperbolic orbits intersect transversally. Then Ξ_1 and Ξ_2 are Baire subsets of $C^\infty(M, (0, \infty))$.*

The question is now if finite energy foliations exist for given data (M, λ, J). The answer to this question in general is not known. However, as we will see, we have existence for $M = S^3$ and generic contact forms $\lambda = f\lambda_0$, where λ_0 is the standard contact form and $f \in \Xi_1$, provided J is generic as well. In the S^3-case it makes sense to impose more conditions on the finite energy foliation.

First of all one needs to define some index $\mu(x, T)$ for a non-degenerate periodic solutions (x, T). This index, the so-called Conley-Zehnder index, [14], is some kind of Morse index for a periodic orbit of a Hamiltonian system. In our low-dimensional case the Conley-Zehnder index can be interpreted as an integer-measure of how orbits infinitesimally close to a periodic orbit twist around it with respect to some natural framing, see [42] for a detailed discussion.

Next one defines another index for a finite energy surface by

$$\operatorname{ind}(\tilde{u}) = \mu(\tilde{u}) - \chi(S) + \sharp\Gamma,$$

where $\chi(S)$ is the Euler characteristic of the underlying closed Riemann surface, $\sharp\Gamma$ is the number of punctures, and $\mu(\tilde{u}) = \mu^+ - \mu^-$ is the total Conley-Zehnder index, which is computed as follows. The number $\mu^\pm$ is the sum of the Conley-Zehnder indices of the periodic orbits associated to the positive and negative punctures, respectively.

The index $\operatorname{ind}(\tilde{u})$ has an interpretation as a Fredholm index, describing the dimension of the moduli space of nearby finite energy surfaces, having the same topological type and number of punctures, which are allowed to move as well as the complex structure on S in Teichmüller space, see [49]. In the following we shall call a solution

$$\tilde{u} : S^2 \setminus \Gamma \to \mathbf{R} \times M$$

of the non-linear Cauchy-Riemann equation with finite (but nontrivial) energy having only non-removable punctures a finite energy sphere. If $\Gamma = \{\infty\}$ we call it a finite energy plane.

DEFINITION 4.4 *Let $\lambda = f\lambda_0$ be a non-degenerate contact form on S^3 and J a complex multiplication for ξ_0. A stable finite energy foliation for (S^3, λ, J) is a finite energy foliation with the following properties:*

1. *Every leaf of the foliation is the image of an embedded finite energy sphere.*
2. *For every leaf the asymptotic limits are simply covered, their Conley-Zehnder indices are contained in $\{1, 2, 3\}$ and they have self-linking number -1.* [4]
3. *Every leaf has precisely one positive puncture, but an arbitrary number of negative punctures.*
4. *For every leaf F, parametrized by a finite energy sphere $\tilde{u}$, which is not a fixed point for the $\mathbf{R}$-action, we have $ind(\tilde{u}) \in \{1, 2\}$.*

Figure 8 on the next page shows an example.
The following result gives the existence of finite energy foliations.

THEOREM 4.5 *For every choice of $f \in \Xi_1$ there exists a Baire set of admissible complex multiplications J admitting a stable finite energy foliation $\mathcal{F}$ of $(S^3, f\lambda_0, J)$.*

We shall not give a proof of the results concerning finite energy foliations in this overview, but refer the reader to the forthcoming paper [53].

Given a stable finite energy foliation of S^3, one can show that the projected surfaces establish a singular foliation of S^3 which gives a smooth foliation transverse to the flow in the complement of a finite number of distinguished periodic orbits.

Using this system of surfaces one can prove, [53]:

THEOREM 4.6 *Let $f \in \Xi_2$. Then the Reeb flow of X_λ on S^3 associated to $\lambda = f\lambda_0$ has the following properties.*

- *Either X_λ has precisely two geometrically distinct periodic orbits or infinitely many.*
- *If X_λ does not admit a disk-like global surface of section there exists a hyperbolic periodic orbit with orientable stable manifold and a homoclinic orbit converging in forward and backward time to the hyperbolic orbit. In particular there are infinitely many geometrically distinct periodic orbits and the topological entropy of the flow is positive.*

This gives the following corollary.

COROLLARY 4.7 *Let $f \in \Xi_2$. If the associated Reeb flow admits a periodic orbit (x, T), with T the minimal period, so that $x_T : \mathbf{R}/(T\mathbf{Z}) \to S^3$ is knotted, then there exist infinitely many geometrically distinct periodic orbits.*

[4] Presumably one can also require the asymptotic limits to be unknotted. However, our existence result Theorem 4.5 so far does not give this additional property.

Figure 8: *The figure shows the trace of the projection of a finite energy foliation on a two-dimensional plane. Here we have two spanning orbits E_1 and E_2 which are elliptic and a hyperbolic one denoted by H. Moreover the foliation contains planes and cylinders. The dashed lines are the traces of the stable and unstable manifold of the hyperbolic orbit H. We assume the non-generic situation that they precisely match up creating several invariant sets. The dotted lines are periodic orbits for the Reeb vector field. The fat lines represent rigid pieces of the finite energy foliation. Namely two cylinders and two planes. The three-sphere is viewed as $\mathbf{R}^3 \cup \{\infty\}$.*

It is worthwhile to give some ideas about the proof of Corollary 4.7. Given $\lambda = f\lambda_0$ take a generic J and the associated stable finite energy foliation $\mathcal{F}$ for (S^3, λ, J). Assume that the $\mathbf{R}$-action has precisely one fixed point. In this case we have a disk-like surface of section $\mathcal{D}$ and a return map

$$\Psi : \dot{\mathcal{D}} \to \dot{\mathcal{D}},$$

which preserves the area form $d\lambda|\mathcal{D}$. This map has a fixed point as a consequence of Brouwer's translation theorem. Recall that the translation theorem asserts that an orientation preserving homeomorphism h of $\mathbf{R}^2$ either has a fixed point or there exists a non-empty open set U such that $U \cap h^j(U) = \emptyset$ for all $j \in \{1, 2, \ldots\}$. Clearly all $h^j(U)$ and $h^k(U)$ are mutually disjoint for $j \neq k$. If in our case Ψ does not have a fixed point we immediately obtain a contradiction to the fact that Ψ preserves area. Removing this fixed point we obtain an area preserving self-map of the open annulus, which by a striking result due to Franks, [29], has the following property:

Theorem 4.8 (Franks) *Let Ψ be an area- and orientation-preserving self-map of the open annulus. If Ψ admits a periodic point, then it admits infinitely many periodic points.*

In order to finish the argument for the corollary we may assume arguing indirectly that there are precisely two periodic orbits. In that case both are unknotted

Figure 9: *The figure shows the situation if there is a disk-like surface of section, but only two periodic orbits.*

and mutually linked, as depicted in Figure 9. However, since we have one knotted orbit this is impossible.

There are also results without any genericity assumption. If $M \subset \mathbf{R}^4$ is a compact energy surface enclosing a strictly convex domain, then one can show by methods similar to those outlined above that there exists a global disk-like surface of section. More precisely, see [48],

THEOREM 4.9 *The Hamiltonian flow on a a sphere-like energy surface in $\mathbf{R}^4$, bounding a strictly convex domain admits a global disk-like surface of section. In particular it has precisely two geometrically distinct periodic orbits or infinitely many.*

5 Topology and Reeb dynamics

After the previous results and discussions one might wonder, if it is possible to use the theory of finite energy surfaces and some knowlegde of the Reeb dynamics in order to learn something about the topology of the underlying manifold. There has been not much research in that direction, but the results so far indicate that there are some nontrivial connections.

We begin by showing that tight contact forms feel the topology. Let M be a closed three manifold. For every tight contact form λ denote by $[\lambda]$ the infimum of all periods T of contractible periodic orbits (x, T) for X_λ. [5] For a closed oriented surface $F \subset M$ denote by $v_\lambda(M)$ the number

$$v_\lambda(F) = \frac{\frac{1}{2}\int_F |d\lambda|}{[\lambda]}.$$

This is the normalized positive area of F.[6] Then define the virtual area of F by

$$v(F) = \inf_{\lambda \in \mathcal{T}} v_\lambda(F).$$

[5] If there are no contractible periodic orbits we take the infimum over the empty set leading to $[\lambda] = \infty$. For simple geometrical reasons we always have $[\lambda] > 0$.

[6] Obviously $\int_F d\lambda = 0$, so that the positive area and the negative area cancel each other.

Here $\mathcal{T}$ is the collection of all tight contact forms on M.

One has the following result, see [40].

Theorem 5.1 *Assume M is a closed orientable three-manifold and F an embedded sphere. If $v(F) < 1$ then F is contractible in M.*

If the Poincaré conjecture holds one can show that $v(F) < 1$ implies $v(F) = 0$ and even that $v(F) = 0$ if and only if F is contractible. The criterion is extremely sharp. For $F = \{point\} \times S^2$ in $M = S^1 \times S^2$ we have $v(F) = 1$.[7]

The next result shows that we are even sometimes able to recover the topology of the space. For more general results see [46, 47].

Theorem 5.2 *Assume that λ is a contact form on the closed three-manifold M, so that the periodic orbits of the associated Reeb vector field are all non-degenerate. Assume that there exists an embedded disk $\mathcal{D}$ in M so that the boundary $\partial\mathcal{D}$ is a periodic orbit of minimal period T_0, say, and $\mathcal{D} \setminus \partial\mathcal{D}$ is transverse to the flow. Then, if all periodic orbits with periods $T \leq T_0$ are elliptic or hyperbolic with non-orientable stable manifold, necessarily M is diffeomorphic to S^3 and the contact form λ is tight.*

Now leaving firm grounds one might foresee some of the possible developments in contact geometry and topology as follows. Assume a contact form λ on a closed three-manifold M is given. Fixing an admissible complex multiplication for the underlying contact structure ξ gives an almost complex struxcture $\tilde{J}$ for $\mathbf{R} \times M$. Studying the finite energy surfaces for $\tilde{J}$ will lead[8] to some Floer type homology theory, called contact homology, build on a $\mathbf{Z}_2$-graded algebra generated by the periodic orbits. The analytical difficulties comprise those familiar in the Arnold conjectures, [30, 63, 64]. The underlying techniques are those from [40, 50, 49, 51, 48, 52]. As it turns out, contact homology only depends on the underlying co-oriented contact structure ξ. This theory can be carried out in any (odd) dimension. Symplectic cobordisms compatible with the contact structures induce morphisms in this theory.

Focusing now on dimension three the following can be said. The contact homology for overtwisted contact structures is presumably trivial, and, if ξ is tight, an interesting invariant for (M, ξ). Given a Legendrian knot, i.e. a knot with tangent space contained in ξ, certain surgeries are possible to lead to new tight contact manifolds. It is important to understand how the contact groups change. Of course, it is necessary to introduce a contact homology group for

[7]As a parenthetical remark we observe that for every tight contact form

$$v(F) \cdot [\lambda] \leq \frac{1}{2} \int_F |d\lambda|.$$

In case, there exists an embedded non-contractible sphere, which always holds if $\pi_2(M) \neq \{0\}$ by the sphere theorem, we have that $v(F) \geq 1$. Therefore the inequality implies the existence of a contractible periodic orbit.

[8]The details for such a theory are formidable and are just being carried out by Y. Eliashberg and H. Hofer.

Legendrian knots, [24], which would be based on Arnold's chord problem, [4] and would have to generalize [13].

Since the contact homology for (M,ξ) should be computable for every generic contact form inducing ξ it will be important to develop methods to simplify the contact form by eliminating short periodic orbits for the Reeb vector field, which algebraically should not be there.

Thirdly, some of the finite energy surfaces occurring for a given contact form, might be used for finite energy foliations, which lead to generalizations of open book decompositions, but indeed carrying more structure.

It is feasible that some program as out-lined above will be useful for studying the topology of three-dimensional manifolds. There is, of course, no doubt that this program leads in any case to a deeper understanding of the dynamics of Reeb vector fields. This is particularly interesting, since we also obtain new tools for studying geodesic flows on surfaces.

6 Relationship to other areas

In a nutshell one can say that study of the Reeb dynamics or certain aspects of it is closely related to be able to count and handle holomorphic curves. How to use holomorphic spheres in order to prove cases of the Weinstein conjecture was shown by Hofer and Viterbo, [45]. Of course meanwhile there are very well-developed methods for counting holomorphic curves in a systematic way, leading to the Gromov-Witten invariants, see [30, 63]. That these invariants can be effectively used for proving certain cases of the Weinstein conjecture has been shown recently in [65].

Theorem 6.1 *Let (V,ω) be any compact symplectic manifold. Then the Weinstein conjecture holds for every hypersurface of contact type in $(\mathbf{C}\times V,\omega_{\mathbf{C}}\oplus\omega)$.*

In dimension four Gromov-Witten invariants are closely related to Seiberg-Witten theory by the important results of Taubes, see [82, 83]. These results guarantee that in a four-dimensional symplectic manifold certain two-dimensional cohomology classes can be represented by a holomorphic curve.

How one can bring all these theories nicely together has been demonstrated by Weimin Chen, [10].

Theorem 6.2 (Weimin Chen) *Let $M\subset(V,\omega)$ be a compact hypersurface of contact type in a closed symplectic four-manifold with $b_2^+(V)\geq 2$. Let λ be a contact form on M, so that $d\lambda=\omega|M$. Assume M carries the orientation induced by $\lambda\wedge d\lambda$. Then the first Chern class of the induced contact structure $\xi=kern(\lambda)$ equipped with a complex structure compatible with $d\lambda$ is Poincaré dual to a finite union of periodic orbits on M oriented by $-\lambda$. In particular if $c_1(\xi)\neq 0$ there has to be a periodic orbit.*

The key ingredient is a the following theorem of Taubes.

Theorem 6.3 (Taubes) *Let (V,ω) be a closed symplectic four-manifold with $b_2^+(V)\geq 2$ and a nontrivial canonical bundle K. Then for a generic ω-compatible* *almost complex structure J, the Poincaré dual to $c_1(K)$ is represented by the fundamental class of an embedded J-holomorphic curve Σ in V (not necessarily connected).*

This result follows from the relationship between Seiberg-Witten and Gromov-Witten invariants and the nontriviality of the Seiberg-Witten invariants for closed symplectic manifolds, see [82, 83].

The proof of Theorem 6.2 has a certain number of technical ingredients. Nevertheless a proof by pictures gives an idea.

Figure 10: *Stretching of a holomorphic curve*

The compact hypersurface M sits inside V and has an open neigborhood $[-\varepsilon, \varepsilon] \times M$ with symplectic structure $d(e^t\lambda)$. We take an almost complex structure $\hat{J}_N$ compatible with ω, which behaves on $[-\varepsilon, \varepsilon] \times M$ in such a way that in suitable coordinates the neigbhorhhood looks like $[-N, N] \times M$ equipped with $\tilde{J}$[9] Taubes' result guarantees for every N a holomorphic curve C_N. The additional information guarantees certain bounds on the area as well as on the genus. In the limit $N \to \infty$ the curve converges near $\{0\} \times M$ to some cylinders over periodic orbits.

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