

SPECIAL SUBVARIETIES IN MUMFORD-TATE VARIETIES

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ABSTRACT. Let $X = \Gamma \backslash D$ be a Mumford-Tate variety, i.e., a quotient of a Mumford-Tate domain $D = G(\mathbb{R})/V$ by a discrete subgroup Γ . Mumford-Tate varieties are generalizations of Shimura varieties. We define the notion of a special subvariety $Y \subset X$ (of Shimura type), and formulate necessary criteria for Y to be special. Our method consists in looking at finitely many compactified special curves C_i in Y , and testing whether the inclusion $\bigcup_i C_i \subset Y$ satisfies certain properties. One of them is the so-called relative proportionality condition. In this paper, we give a new formulation of this numerical criterion in the case of Mumford-Tate varieties X . In this way, we give necessary and sufficient criteria for a subvariety Y of X to be a special subvariety of Shimura type in the sense of the André-Oort conjecture. We discuss in detail the important case where $X = A_g$, the moduli space of principally polarized abelian varieties.

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1 INTRODUCTION

Griffiths domains [4] are flag domains, i.e., quotients of the form $D = G(\mathbb{R})/V$, where G is a certain reductive algebraic group defined over \mathbb{Q} and V a compact stabilizer subgroup. Griffiths domains parametrize pure Hodge structures of given weight and Hodge numbers. Any moduli space \mathcal{M} of smooth, projective varieties induces, after a choice of cohomological degree and a base point, a period map

$$\mathcal{P} : \mathcal{M} \rightarrow \Gamma \backslash D,$$

where Γ is the monodromy group, i.e., the image of the fundamental group of \mathcal{M} in $G(\mathbb{R})$, a finitely generated, discrete subgroup.

In general, the period map \mathcal{P} is not surjective, but has image contained in quotients of so-called Mumford-Tate domains by discrete subgroups, see [4, Chap. 15] or [7]:

THEOREM 1.1. *After possibly replacing \mathcal{M} by a finite, étale cover, the period map \mathcal{P} factors as*

$$\mathcal{P} : \mathcal{M} \longrightarrow \Gamma^{nc} \backslash D^{nc} \times \Gamma^c \backslash D^c \times D^f,$$

into a product of quotients of domains of non-compact, compact or flat (i.e., constant) type. Here D^\bullet denotes a domain of the respective type. The composition with the third projection is constant. In addition, for each $x_1 \in \Gamma^{nc} \backslash D^{nc}$ and $x_3 \in D^f$, one has that $\text{Im}(\mathcal{P}) \cap (x_1 \times \Gamma^c \backslash D^c \times x_3)$ is finite.

Recall that a domain is by definition of non-compact type (resp. compact type, resp. flat type) if its universal cover is a product of non-compact irreducible spaces (resp. product of compact irreducible spaces, resp. is a euclidean space). Theorem 1.1 asserts that the "non-compact part" of the period map is the essential one. The (derived) Mumford-Tate group of the Hodge structure of a general element in \mathcal{M} contains the algebraic monodromy group, i.e., the \mathbb{Q} -Zariski closure of the topological monodromy group, as a normal subgroup by a theorem of Y. André [4, Prop. 15.8.5].

IN THE REST OF THIS PAPER, WE WILL ASSUME THAT G IS OF NON-COMPACT TYPE, \mathbb{Q} -SIMPLE AND ADJOINT. It is not difficult to reduce to this case. Only in rare cases, D itself is Hermitian symmetric [4]. In these cases, $\Gamma \backslash D$ is a connected component of a Shimura variety under some arithmetic condition on Γ [5, 10]. An important example is the moduli space $A_g = \Gamma \backslash \mathbb{H}_g$ of principally polarized abelian varieties of dimension g with some level structure induced by Γ . Here \mathbb{H}_g denotes the Siegel upper half space of genus g . Shimura varieties contain distinguished subvarieties which are called special subvarieties. The zero-dimensional special subvarieties are the CM points, i.e., the points corresponding to Hodge structures with commutative Mumford-Tate group. Positive dimensional special subvarieties are more difficult to understand. However, the André-Oort conjecture claims that special subvarieties of Shimura varieties are precisely the loci which are the Zariski closures of sets of CM points. This conjecture has recently attracted a lot of interest, see the work of Edixhoven, Klingler, Pila, Ullmo, Tsimerman, Yafaev and others [6, 9, 13, 14, 16]. In 2015, based on the above works, Tsimerman [16] completed the proof of the André-Oort conjecture for A_g using an averaged version of a conjecture of Colmez proved by Yuan and Zhang [19] and also independently by Andreatta-Goren-Howard-Madapusi in [2].

Our aim is to give sufficient and effective Hodge theoretic criteria for a subvariety of $X = \Gamma \backslash D$ to be a special subvariety.

In [11] and [12], we have studied special subvarieties in Shimura varieties of unitary or orthogonal type. Our method consisted of characterizing special

subvarieties by a relative proportionality principle. The main goal of the present work is to generalize this principle to quotients of Mumford-Tate domains.

1.1 RESULTS IN THE CASE $X = A_g$

For the reader’s convenience, we first study the case where $X = A_g$. Let $A_g = \Gamma \backslash \mathbb{H}_g$ be a smooth model, i.e., we require that Γ is torsion-free. Recall that by \mathbb{H}_g we denote the Siegel upper half space of genus g . We choose a smooth toroidal compactification \overline{A}_g as constructed by Mumford et al. [3, chap. III], such that the boundary $S \subset \overline{A}_g$ is a divisor with normal crossings. We consider a smooth projective subvariety $Y \subset \overline{A}_g$ meeting S transversely and define $Y^0 := Y \cap A_g$. Our results are valid for any compactification \overline{A}_g satisfying these conditions. Throughout this paper we denote subvarieties contained in the locally symmetric part A_g of \overline{A}_g with a superscript 0.

Such a subvariety Y^0 is called special, if it is an irreducible component of a Hecke translate of the image of some morphism $Sh_K(G, X) \rightarrow A_g = Sh_{K(N)}(GSp(2g), \mathbb{H}_g^\pm)$, defined by an inclusion of a Shimura subdatum $(G, X) \subset (GSp(2g), \mathbb{H}_g^\pm)$ together with some compact open subgroup $K \subset G(\mathbb{A}_f)$ such that $K \subset K(N)$. See Section 3 for details about Shimura varieties and special subvarieties.

We look for necessary and sufficient effective criteria, such that Y^0 is a special subvariety with minimal dimension containing a union $\bigcup_{i \in I} C_i^0$ of finitely many special curves C_i . Already in our previous works [11] and [12] we have found a necessary condition for Y^0 to be special, provided a compactified special curve $C \subset \overline{A}_g$ (i.e., compactification of a special curve $C^0 \subset Y^0$) is contained in Y :

DEFINITION 1.2 (Relative Proportionality Condition (RPC)).

Let $C \subset Y \subset \overline{A}_g$ be a compactified irreducible special curve with logarithmic normal bundle $N_{C/Y}$, and 3-step Harder-Narasimhan filtration $0 \subset N_{C/Y}^0 \subset N_{C/Y}^1 \subset N_{C/Y}^2 = N_{C/Y}$ (both notions are explained in Section 4). Then one has the relative proportionality inequality

$$\deg N_{C/Y} \leq \frac{\text{rank}(N_{C/Y}^1) + \text{rank}(N_{C/Y}^0)}{2} \cdot \deg T_C(-\log S_C).$$

If C and Y are special subvarieties, then equality holds.

For curves C on Hilbert modular surfaces or Picard modular surfaces, this condition is only a simple numerical criterion involving intersection numbers, see [11] and [12].

Suppose we are given a finite number of compactified special curves C_i in \overline{A}_g , contained in some irreducible subvariety Y of dimension $\dim(Y) \geq 2$. We assume for simplicity that Y and all C_i intersect the boundary S of A_g transversely. Fix a base point $y \in Y^0 \subset A_g$ contained in the union of all C_i and assume for simplicity that the union $\bigcup_{i \in I} C_i^0$ is connected. Let \mathbb{V} be the local system of weight 1 polarized \mathbb{Q} -Hodge structures over A_g . More

precisely, choose a level $N \geq 3$ structure $A_g^{[N]}$ on A_g . Denote by $f : U \rightarrow A_g$ the universal family of abelian varieties, then $\mathbb{V} = R^1 f_* \mathbb{Q}$ is the local system attached to it. So over each point $y \in A_g$, \mathbb{V}_y is the associated polarized \mathbb{Q} -Hodge structure $H^1(A_y, \mathbb{Q})$ with A_y being the abelian variety corresponding to y . Let $(E, \theta) = (E^{1,0} \oplus E^{0,1}, \theta)$ be the Higgs bundle associated with $\mathbb{V}_{\mathbb{C}}$. The thickening of the Higgs field is the pullback of the Higgs bundle on Y via $\varphi : C \rightarrow Y$:

$$\theta_{C/Y} := \varphi^* \theta : E^{1,0} \rightarrow E^{0,1} \otimes \varphi^* \Omega_Y^1(\log S_Y).$$

We can also consider the thickening in a point $y \in Y$:

$$\theta_{y \in Y} : E_y^{1,0} \rightarrow E_y^{0,1} \otimes \varphi^* \Omega_Y^1(\log S_Y)_y.$$

Consider the k -fold tensor product $(E, \theta)^{\otimes k}$. It decomposes as a direct sum $E^{\otimes k} = \sum_{m+n=k} E^{m,n}$, where $E^{m,n} = \bigoplus E^{m_1, n_1} \otimes \dots \otimes E^{m_k, n_k}$ and the sum

ranges over all (m_i, n_i) with $m_i + n_i = 1, \sum_{i=1}^k m_i = m, \sum_{i=1}^k n_i = n$. The Higgs field, which we continue to denote by θ , decomposes as $\theta : E^{m,n} \rightarrow E^{m-1, n+1} \otimes \Omega_Y^1(\log S_Y)$ with

$$E^{m_1, n_1} \otimes \dots \otimes E^{m_k, n_k} \xrightarrow{\theta} \bigoplus_{i=1}^k E^{m_1, n_1} \otimes \dots \otimes E^{m_i-1, n_i+1} \otimes \dots \otimes E^{m_k, n_k} \otimes \Omega_Y^1(\log S_Y)$$

satisfying the Leibniz rule. In particular, given a base point $y \in \varphi(C) \subset Y$, we get the map

$$\theta_{y \in Y} : E_y^{p,p} \rightarrow E_y^{p-1, p+1} \otimes \Omega_Y^1(\log S_Y)_y.$$

Note that the complex vector space $E_y^{p,p}$ does not have any natural \mathbb{Q} -structure, however since in the fibers $E_y^{p,p} \subset \mathbb{V}_y^{\otimes k} \otimes \mathbb{C}$, we define

DEFINITION 1.3. Let

$$W_{y \in Y} := \{v \in E_y^{p,p} \cap \mathbb{V}_{y, \mathbb{Q}}^{\otimes k} \mid \theta_{y \in Y}(v) = 0\},$$

i.e., the Hodge tensors that are killed by the (infinitesimal) Higgs field in $y \in Y$. Note that if k is not even, then we require that $W_{y \in Y}$ is trivial. The tensors in $W_{y \in Y}$ are called *infinitesimally parallel Hodge tensors*. We define the group H_y to be the largest \mathbb{Q} -algebraic group fixing the vectors in all $W_{y \in Y}$ (for all p).

The spaces $W_{y \in Y}$ (more generally $W_{y \in Y}$ for a finitely generated sublocal system $\mathbb{W} \subset \mathbb{V}^{\otimes}$, see Definition 1.8) and the group H_y will play a crucial role in the sequel, see §7.

For the following definition, note that as remarked above, by the construction of the Higgs bundle associated to a local system, it holds that in the fibers

$E_y^{m,n} \subset \mathbb{V}_y^{\otimes k}$ and so we can do the parallel transport using the connection fiberwise (but the connection associated to the local system does not descend to the graded bundle $E^{m,n}$ in general). Note that it is not in general true that the property of being a (p, p) -class is preserved under the parallel transport.

DEFINITION/REMARK 1.4. Let

$$(E_y^{p,p})_{par} = \{v \in E_y^{p,p} \cap \mathbb{V}_{y,\mathbb{Q}}^{\otimes k} \mid \text{any parallel transport of } v \text{ from } y \text{ to } y' \text{ lies in } E_{y'}^{p,p} \cap \mathbb{V}_{y',\mathbb{Q}}^{\otimes k}\}.$$

Here $y' \in Y$ varies in Y and we say “parallel transport of v from y to y' ”, where we mean more precisely “parallel transport of v along any path from y to y' “. By parallel transporting one also gets $(E_y^{p,p})_{par} \simeq (E_{y'}^{p,p})_{par}$. We define

$$MT(\mathbb{V})_y = \{g \in GL(E_y) \mid g \text{ stabilizes } v \in (E_y^{p,p})_{par} \text{ for all } p \in \mathbb{N}\}.$$

Note that $(E_y^{p,p})_{par}$ is associated to a polarized sublocal system and since it consists of Hodge tensors (i.e., \mathbb{Q} -tensors of type (p, p)), its polarization is definite and so it is unitary and hence $v \in (E_y^{p,p})_{par}$ is killed by the Higgs field $\theta_{y \in Y}$. Hence, by the above definition of the group H_y , we have the inclusion

$$H_y \subset MT(\mathbb{V})_y.$$

On the other hand, if $MT(\mathbb{V}_y)$ is the Mumford-Tate group fixing all Hodge tensors in \mathbb{V}_y , then it trivially holds that $MT(\mathbb{V}_y) \subset H_y$. So in general we have the inclusions

$$MT(\mathbb{V}_y) \subset H_y \subset MT(\mathbb{V})_y.$$

Note that by parallel transport we have an isomorphism $MT(\mathbb{V})_y \simeq MT(\mathbb{V})_{y'}$ (but in general the isomorphism $MT(\mathbb{V}_y) \simeq MT(\mathbb{V}_{y'})$ is not true).

Now we are ready to state our first result:

THEOREM 1.5. *Let Y^0 be a smooth, algebraic subvariety of A_g such that Y^0 has unipotent monodromies at infinity. Let Y be a smooth compactification of Y^0 as above such that Y intersects the boundary S of \overline{A}_g transversely. Suppose Y^0 contains finitely many special curves C_i^0 such that the compactification C_i of C_i^0 is a special curve in Y and that $\bigcup_{i \in I} C_i^0$ is connected. Choose a base point $y \in \bigcup_{i \in I} C_i^0$. Assume the following:*

- (BIG) The \mathbb{Q} -Zariski closure in $G = \mathrm{Sp}(2g)$ of the monodromy representation of $\pi_1(\bigcup_{i \in I} C_i^0, y)$ equals the \mathbb{Q} -Zariski closure of the representation of $\pi_1(Y^0, y)$.*
- (LIE) If $H = H_y$ is the \mathbb{Q} -algebraic group in Definition 1.3 and $H(\mathbb{R})/K$ is the associated period domain, then one has $\dim H(\mathbb{R})/K \leq \dim Y$.*
- (RPC) All compactified special curves C_i satisfy relative proportionality.*

Then, Y^0 is a special subvariety of A_g .

In addition, the proof of the theorem implies that K is a maximal compact stabilizer group, that H is of Hermitian type (i.e., H/K is a Hermitian symmetric space), and in the Hodge decomposition $\mathfrak{h}_{\mathbb{C}} = \mathfrak{h}^{-1,1} \oplus \mathfrak{h}^{0,0} \oplus \mathfrak{h}^{1,-1}$ of the real Lie algebra $\mathfrak{h} = \text{Lie } H(\mathbb{R})$, one has $\mathfrak{h}^{-1,1} = T_{Y^0, y}$ for the holomorphic tangent space of Y^0 at y . In particular, the group H_y does not depend on the base point y in a crucial way.

More can be said about the group $H = H_y$: In fact, in general the holomorphic tangent space $T_{Y^0, y}$ is a subspace of the holomorphic tangent space of H/K at y and $H(\mathbb{R})/K \subset \mathbb{H}_g$ is the smallest Mumford-Tate subdomain which contains y and such that the holomorphic tangent space of $H(\mathbb{R})/K$ at y contains $T_{Y^0, y}$, see Proposition 1.10. Therefore, one always has $\dim H(\mathbb{R})/K \geq \dim Y^0$, and condition (LIE) implies that $\dim H(\mathbb{R})/K = \dim Y^0$.

Theorem 1.5 generalizes previous work in [11] and [12], which was restricted to special subvarieties in unitary or orthogonal Shimura varieties, hence the case of rank ≤ 2 . There are explicit examples of connected cycles $\cup_i C_i^0$ of special curves C_i^0 in A_g for $g \geq 2$, for which the minimal enveloping special subvariety of $\cup_i C_i^0$ is A_g but not smaller. For example two so called Mumford curves (see [18], section 5) in A_4 intersecting at one point. This shows that condition (LIE) is necessary. We saw already above that (RPC) is also necessary. Condition (BIG) is probably not a necessary condition. All three conditions are not independent, but their relations are not fully understood. In the course of the proof, we will see that condition (BIG) together with (RPC) implies that the monodromy group Γ (i.e. the \mathbb{Q} -Zariski closure $\overline{\rho(\pi_1(Y^0, y))}$) is contained in the group $H = H_y$ defined above. We therefore formulate a condition as follows

$$\text{(Mon)} \quad \Gamma \subset H_y$$

See the last section of this introduction for a strategy of the proof of Theorem 1.5.

1.2 RESULTS IN THE CASE OF A MUMFORD-TATE VARIETY $X = \Gamma \backslash D$

Now we turn to the general case. As far as we know, there is no good notion of Hecke operators on Mumford-Tate domains $D = G(\mathbb{R})/V$. In addition, there are no good compactifications of a Mumford-Tate variety $X = \Gamma \backslash D$ known in these cases in general [8].

Therefore, to avoid these two difficulties, by a special curve in X we will denote an étale morphism

$$\varphi^0 : C^0 \longrightarrow X$$

from a Shimura curve C^0 , which is induced from a morphism of algebraic groups $G' \rightarrow G$ defined over \mathbb{Q} , such that a certain Shimura datum for G' defines C^0 . Assume also that we are given a quasi-projective variety $Y^0 \subset X$ containing the image of φ^0 and with a NC smooth compactification Y and let C be a compactification of C^0 . Denote by $S_Y = Y \setminus Y^0$ the boundary divisor, and by $S_C = C \setminus C^0$, so that S_C is the pullback of S_Y to C , and φ^0 extends to a finite map $\varphi : C \rightarrow Y$.

In Section 6, we show that there is a filtration

$$N_{C/Y}^0 \subset N_{C/Y}^1 \subset \cdots \subset N_{C/Y}^s = N_{C/Y}$$

on the logarithmic normal bundle $N_{C/Y}$, induced by the Harder-Narasimhan filtration on $N_{C/X}$. The logarithmic normal bundle $N_{C/Y}$ is defined by the exact sequence

$$0 \rightarrow T_C(-\log S_C) \rightarrow \varphi^* T_Y(-\log S_Y) \rightarrow N_{C/Y} \rightarrow 0.$$

The relative proportionality condition can be stated as:

DEFINITION 1.6 (Relative Proportionality Condition (RPC)).

The curve $\varphi : C \rightarrow Y$ satisfies the *relative proportionality condition* (RPC) if the slope inequalities

$$\mu(N_{C/Y}^i/N_{C/Y}^{i-1}) \leq \mu(N_{C/X}^i/N_{C/X}^{i-1}), \text{ for } i = 0, \dots, s$$

are equalities. The sheaves $N_{C/X}^i$ are properly defined in Section 6. The integer s depends on C and X . Summing up these inequalities, yield the relative proportionality inequality

$$\deg N_{C/Y} \leq r(C, Y, X) \cdot \deg T_C(-\log S_C),$$

where $r(C, Y, X) \in \mathbb{Q}$ is a rational number depending on C , Y and X , and hence on G . If C and Y are special subvarieties, then equality holds.

Let $\mathfrak{g} = \text{Lie}(G)$ be the Lie algebra of G . We have a weight zero Hodge structure on \mathfrak{g} ,

$$\mathfrak{g} = \bigoplus_p \mathfrak{g}^{-p,p}.$$

DEFINITION 1.7. We denote by T_X^h the holomorphic, horizontal tangent bundle to X [4, Sec. 12.5]. That is, T_X^h is the homogenous bundle on X associated to $\mathfrak{g}^{-1,1}$.

Now we prove the analogue of Theorem 1.5 for Mumford-Tate varieties. We will assume that Y^0 is a *horizontal subvariety* of X , i.e., that T_{Y^0} is contained in the horizontal tangent bundle T_X^h . For the following theorem, we need also to generalize the notion of special subvariety of Shimura type of a general Mumford-Tate variety X . This will be done in section 5. Let \mathbb{V} be a local system of polarized \mathbb{Q} -Hodge structures over X . So over each point $y \in X$, \mathbb{V}_y is a polarized \mathbb{Q} -Hodge structure of some given weight. Choose any finitely generated, sublocal system $\mathbb{W} \subset \mathbb{V}^{\otimes}$ of even weight $2p$ and defined over \mathbb{Q} , where \mathbb{V}^{\otimes} is the full tensor algebra generated by tensor powers of \mathbb{V} and its dual. We denote the fiber of $\mathbb{W}_{\mathbb{Q}}$ over y by $W_{y,\mathbb{Q}}$. Let (E, ϑ) be the Higgs bundle corresponding to $\mathbb{W}_{\mathbb{C}}$.

Assume now that $\varphi : C \rightarrow Y$ compactifies $C^0 \rightarrow Y^0 \rightarrow X$ with C^0 a special curve in X . If $\varphi : C \rightarrow Y$ satisfies (RPC), then we have a decomposition:

$$N_{C/Y} = N_{C/Y}^0 \oplus N_{C/Y}^1/N_{C/Y}^0 \oplus \cdots \oplus N_{C/Y}^i/N_{C/Y}^{i-1} \oplus \cdots \oplus N_{C/Y}^s/N_{C/Y}^{s-1}.$$

Now let

$$\theta_{y \in Y} := E_y^{p,p} \rightarrow E_y^{p-1,p+1} \otimes \Omega_Y^1(\log S_Y)|_y$$

be the thickening of the Higgs field along C , as explained in section 1.2, with splitting

$$E_y^{p-1,p+1} \otimes \Omega_Y^1(\log S_Y)|_y \cong E_y^{p-1,p+1} \otimes \left(\Omega_C^1(\log S_C)|_y \oplus N_{C/Y}^V|_y \right).$$

Although the complex vector space $E_y^{p,p}$ does not have any natural \mathbb{Q} -structure, we define as in the Definition 1.3.

DEFINITION 1.8. Under these assumptions, we define a complex vector space in analogy with Definition 1.3

$$W_{y \in Y} := \{t \in E_y^{p,p} \cap \mathbb{W}_{y,\mathbb{Q}} \mid \theta_{y \in Y}(t) = 0\}.$$

As in the case of A_g , the tensors in $W_{y \in Y}$ are called infinitesimally parallel Hodge tensors. We define the group H_y to be the largest \mathbb{Q} -algebraic group fixing the vectors in all $W_{y \in Y}$.

With the above definition, the condition (Mon) in the general case is formulated exactly as in the previous section, namely the monodromy group Γ (i.e. the \mathbb{Q} -Zariski closure $\rho(\pi_1(Y^0, y))$) is contained in the group $H = H_y$ defined above.

$$\text{(Mon)} \quad \Gamma \subset H_y$$

THEOREM 1.9. *Let $X = \Gamma \backslash D$ be a Mumford-Tate variety associated to the Mumford-Tate group G . Let Y^0 be a smooth, horizontal algebraic subvariety of X that has unipotent monodromies at infinity. Moreover let Y be a NC smooth compactification of Y^0 . Suppose that there exists a finite collection of special curves $\varphi_i^0 : C_i^0 \rightarrow Y^0$ such that $\bigcup_{i \in I} C_i^0$ is connected with compactifications $\varphi_i : C_i \rightarrow Y$ as above and NC divisors $S_{C_i} = C_i \setminus C_i^0$. Choose a base point $y \in \bigcup_{i \in I} C_i^0$. Assume the following:*

(BIG) *The \mathbb{Q} -Zariski closure in the Mumford-Tate group G of the monodromy representation of $\pi_1(\bigcup_{i \in I} C_i^0, y)$ equals the \mathbb{Q} -Zariski closure of the representation of $\pi_1(Y^0, y)$.*

(LIE) *If $H = H_y$ is the \mathbb{Q} -algebraic group of Definition 1.8 and K a compact subgroup such that $H(\mathbb{R})/K \subset D$ is the period domain associated to H , then one has $\dim H(\mathbb{R})/K \leq \dim Y$.*

(RPC) *All compactified special curves C_i satisfy relative proportionality. Then, Y^0 is a special subvariety of X of Shimura type (see Definition 5.1).*

In addition, as in the case of A_g , the proof of the theorem implies that the group H essentially does not depend on y , K is a maximal compact stabilizer group, that H is of Hermitian type, and in the Hodge decomposition $\mathfrak{h}_{\mathbb{C}} = \mathfrak{h}^{-1,1} \oplus \mathfrak{h}^{0,0} \oplus \mathfrak{h}^{1,-1}$ of the real Lie algebra $\mathfrak{h} = \text{Lie } H(\mathbb{R})$, one has $\mathfrak{h}^{-1,1} = T_{Y^0,y}$ for the holomorphic tangent space of Y^0 at y . See Proposition 1.10 and the paragraph following this proposition for the above claims.

The (RPC) condition implies that the above filtration is in fact the Harder-Narasimhan filtration on $N_{C/Y}$.

STRATEGY OF THE PROOF

The proof of both theorems is based on the following observations:

PROPOSITION 1.10. *Let $X = \Gamma \backslash D$ be a Mumford-Tate variety associated to the Mumford-Tate group G . Let Y^0 be a smooth, horizontal algebraic subvariety of X such that Y^0 has unipotent monodromies at infinity. Assume the conditions (Mon) and (LIE). Then Y^0 is special.*

Proof. Let Γ be the image of $\pi_1(Y^0, y)$ under ρ in G . The map of universal covers $\widetilde{Y^0} \rightarrow G(\mathbb{R})/V$ factors through $\widetilde{Y^0} \rightarrow H(\mathbb{R})^+/K \hookrightarrow G(\mathbb{R})/V$ and using condition (Mon), the period map may be viewed as a map

$$Y^0 \xrightarrow{\mathcal{P}} Z^0 = \Gamma \backslash H(\mathbb{R})^+/K.$$

By condition (LIE), one has $Y^0 = Z^0$ for dimension reasons. Since Y^0 is horizontal by assumption, it follows that Y^0 is special. \square

Note that the tangent space of the image of the above period map \mathcal{P} lies in the horizontal subspace of the tangent space of $H(\mathbb{R})/K$. By the condition (LIE), the tangent space of the image of the period map is equal to the tangent space of $H(\mathbb{R})/K$. Hence the tangent space of $H(\mathbb{R})/K$ itself is horizontal. Therefore H is of Hermitian type and in particular K is maximal compact.

Using this Proposition, the proofs of Theorem 1.5 and Theorem 1.9 are reduced to the proof of the following Theorem:

THEOREM 1.11. *Let $X = \Gamma \backslash D$ be a Mumford-Tate variety associated to the Mumford-Tate group G . Let Y^0 be a smooth, horizontal algebraic subvariety of X such that Y^0 has unipotent monodromies at infinity. Then, conditions (BIG) and (RPC) imply condition (Mon).*

The condition (BIG) may be replaced by other conditions: for example, one may require that there is an integral linear combination $\sum_{i \in I} a_i C_i$ which deforms in X and fills X out. In [12], we showed that this assumption implies condition (BIG) as well. In this light, we pose the following

CONJECTURE 1.12. *Suppose an irreducible Mumford-Tate variety X associated to G contains (infinitely many) special curves. Then, condition (BIG) holds*

for X , i.e., there are finitely many compactified special curves C_i in X , such that the \mathbb{Q} -Zariski closure of the monodromy representation of $\pi_1(\bigcup_{i \in I} C_i^0, y)$ is equal to G .

In addition, one wants to find an effective bound of the number of special curves needed. Conjecture 1.12 is known to be true in the case where $G = SO(2, n)$ and $G = SU(1, n)$ for $n \geq 1$, see [12, Remark 3.7]. However, it appears to be open even in the case $G = Sp_{2g}$ for large g . It may be possible to solve this conjecture by looking at one special curve C^0 containing a CM-point y and taking finitely many Hecke translates of C^0 which fix the point y .

Theorem 1.11 will be proved in the last section. In the sections before, we recall the notions of special subvarieties and explain the condition (RPC).

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2 MUMFORD-TATE GROUPS AND HODGE CLASSES

For any \mathbb{Q} -algebraic group M , we denote by $M_{\mathbb{R}}$ the associated \mathbb{R} -algebraic group. Let V be a \mathbb{Q} -Hodge structure with underlying \mathbb{Q} -vector space also denoted by V . This corresponds to a real representation

$$h : \mathbb{S} \longrightarrow GL(V_{\mathbb{R}})$$

of the Deligne torus $\mathbb{S} = \text{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m$.

DEFINITION 2.1. The (large) Mumford-Tate group $MT(V)$ of V is the smallest \mathbb{Q} -algebraic subgroup of $GL(V)$ such that $MT(V_{\mathbb{R}})$ contains the image of h . The (special) Mumford-Tate group, or Hodge group, $Hg(V) = SMT(V)$ is the smallest \mathbb{Q} -algebraic subgroup of $SL(V)$ such that $SMT(V_{\mathbb{R}})$ contains the image of the subgroup $\mathbb{S}^1 \subset \mathbb{S}$.

Depending on the context, we will use both groups under the general name Mumford-Tate group. If one looks at all Hodge classes in $V^{\otimes i} \otimes V^{\vee \otimes j}$ for all (i, j) , then the special Mumford-Tate group $SMT(V)$ is precisely the largest \mathbb{Q} -algebraic subgroup $G \subset Sp(2g)$ fixing all Hodge classes in such tensor products.

EXAMPLE 2.2. Let us look at Hodge structures of weight 1. We fix a level N structure $A_g^{[N]}$ on A_g with $N \geq 3$. Therefore, there is a universal family $f : U \rightarrow A_g$ over A_g . Let $\mathbb{V} = R^1 f_* \mathbb{Q}$ be the natural \mathbb{Q} -local system of weight one on A_g . We denote by

$$\mathbb{V}_{\mathbb{C}}^{\otimes} = \bigoplus_{i,j} \mathbb{V}_{\mathbb{C}}^{\otimes i} \otimes \mathbb{V}_{\mathbb{C}}^{\vee \otimes j}$$

the full tensor algebra. This is an infinite direct sum of polarized local systems, where each summand $\mathbb{V}_{\mathbb{C}}^{\otimes i} \otimes \mathbb{V}_{\mathbb{C}}^{\vee \otimes j}$ carries a family of Hodge structures of weight $i - j$. A Hodge class in $\mathbb{V}_{\mathbb{C}}^{\otimes}$ is a flat section in some finite dimensional subsystem of $\mathbb{V}_{\mathbb{C}}^{\otimes}$ defined over \mathbb{Q} and corresponding fiberwise to a (p, p) -class.

3 SPECIAL SUBVARIETIES IN A_g

Let us recall some useful notation concerning Shimura varieties and their special subvarieties.

DEFINITION 3.1 (Shimura datum). A Shimura datum is a pair (G, X) consisting of a connected, reductive algebraic group G defined over \mathbb{Q} and a $G(\mathbb{R})$ -conjugacy class $X \subset \text{Hom}(\mathbb{S}, G_{\mathbb{R}})$ such that for all (i.e., for some) $h \in X$,

- (i) The Hodge structure on $\text{Lie}(G)$ defined by $\text{Ad} \circ h$ is of type $(-1, 1) + (0, 0) + (1, -1)$.
- (ii) The involution $\text{Inn}(h(i))$ is a Cartan involution of $G_{\mathbb{R}}^{\text{ad}}$.
- (iii) The adjoint group G^{ad} does not have factors defined over \mathbb{Q} onto which h has a trivial projection.

The connected components of X form $G(\mathbb{R})^+$ -conjugacy classes. The weight cocharacter $h \circ w : \mathbb{G}_{m, \mathbb{C}} \rightarrow G_{\mathbb{C}}$ does not depend on the choice of h .

For a compact open subgroup $K \subset G(\mathbb{A}_f)$, we consider the double coset space $Sh_K(G, X) = G(\mathbb{Q}) \backslash (X \times G(\mathbb{A}_f)/K)$. We denote an element of this set by $[x, aK]$ ($x \in X, aK \in G(\mathbb{A}_f)/K$). The Shimura variety associated to the above Shimura datum is defined as $Sh(G, X) = \varprojlim Sh_K(G, X)$. Note that $Sh_K(G, X)$ can be recovered from $Sh(G, X)$ as the quotient modulo K by the theory of Shimura varieties. An element $\gamma \in G(\mathbb{A}_f)$ defines a map $Sh_{K'}(G, X) \rightarrow Sh_K(G, X), [x, aK'] \mapsto [x, a\gamma K]$ called the *Hecke translate* for $K' \subset \gamma K \gamma^{-1}$. This map induces a (right) action of $G(\mathbb{A}_f)$ on $Sh(G, X)$. We refer to [10] for an accessible reference concerning Shimura varieties.

A morphism of Shimura data $(M, X') \rightarrow (G, X)$ is a homomorphism $M \rightarrow G$ of algebraic groups sending X' to X . A morphism of Shimura varieties $Sh(M, X') \rightarrow Sh(G, X)$ is an inverse system of regular maps of algebraic varieties compatible with the action of $G(\mathbb{A}_f)$.

Denote by $(GSp(2g), \mathbb{H}_g^{\pm})$ the Shimura datum in the sense of Deligne [5] defining $A_g = A_g^{[N]}$ with level structure given by the compact open subgroup $K(N)$ of $GSp(2g)(\mathbb{A}_f)$.

DEFINITION 3.2 (Special Subvarieties). A special subvariety of A_g is a geometrically irreducible component of a Hecke translate of the image of some morphism $Sh_K(G, X) \rightarrow A_g = Sh_{K(N)}(GSp(2g), \mathbb{H}_g^{\pm})$, which is defined by an inclusion of a Shimura subdatum $(G, X) \subset (GSp(2g), \mathbb{H}_g^{\pm})$ together with some compact open subgroup $K \subset G(\mathbb{A}_f)$ such that $K \subset K(N)$.

In other words, by the above notation, there is a sequence

$$\begin{aligned} Sh(G, X)_{\mathbb{C}} &\longrightarrow Sh(GSp(2g), \mathbb{H}_g^{\pm})_{\mathbb{C}} \xrightarrow{\gamma} Sh(GSp(2g), \mathbb{H}_g^{\pm})_{\mathbb{C}} \\ &\xrightarrow{\text{quot}} A_g = Sh_{K(N)}(GSp(2g), \mathbb{H}_g^{\pm}) \end{aligned}$$

where $\gamma \in G(\mathbb{A}_f)$ and the second map is the Hecke operator defined above.

Special subvarieties are totally geodesic subvarieties with respect to the natural Riemannian (Hodge) metric, i.e., geodesics which are tangent to a special subvariety stay inside. In fact, there is almost an equivalence by a result of Abdulali [1] and Moonen:

PROPOSITION 3.3. *An irreducible algebraic subvariety of A_g is special if and only if it is totally geodesic and contains a CM point.*

Proof. See Theorem 6.9.1 in Moonen [10]. □

4 RELATIVE PROPORTIONALITY IN A_g

Consider a non-singular projective curve C and an embedding

$$\varphi : C \hookrightarrow Y \hookrightarrow \overline{A}_g,$$

where $Y \subset \overline{A}_g$ is a smooth projective subvariety as in the introduction. We denote by $C^0 := \varphi^{-1}(Y^0) \neq \emptyset$ the "open" part, where $Y^0 = Y \cap A_g$. Assume that C^0 is a special curve in the following. Let S_C and S_Y be the intersections of C and Y with S . We assume overall that such intersections are transversal.

The logarithmic normal bundles of C in Y and \overline{A}_g are defined by the exact sequences

$$\begin{aligned} 0 \rightarrow T_C(-\log S_C) \rightarrow T_{\overline{A}_g}(-\log S) \rightarrow N_{C/\overline{A}_g} \rightarrow 0, \\ 0 \rightarrow T_C(-\log S_C) \rightarrow T_Y(-\log S_Y) \rightarrow N_{C/Y} \rightarrow 0. \end{aligned}$$

Let $N_{C/Y}^{\bullet}$ be the Harder-Narasimhan filtration on the logarithmic normal bundle N_{C/\overline{A}_g} intersected with $N_{C/Y}$. The following definition was given in [12, Def. 1.4].

DEFINITION 4.1 (Relative Proportionality Condition (RPC)).

The map $\varphi : C \hookrightarrow Y$ satisfies the relative proportionality condition (RPC), if the slope inequalities

$$\mu(N_{C/Y}^i/N_{C/Y}^{i-1}) \leq \mu(N_{C/\overline{A}_g}^i/N_{C/\overline{A}_g}^{i-1}), \quad i = 0, 1, 2$$

are equalities. For the slopes, one gets by [12]:

$$\begin{aligned} \mu(N_{C/\overline{A}_g}^2/N_{C/\overline{A}_g}^1) &= 0, \\ \mu(N_{C/\overline{A}_g}^1/N_{C/\overline{A}_g}^0) &= \frac{1}{2} \deg T_C(-\log S_C), \\ \mu(N_{C/\overline{A}_g}^0) &= \deg T_C(-\log S_C). \end{aligned}$$

Hence, we obtain a set of inequalities

$$\begin{aligned}\mu(N_{C/Y}^2/N_{C/Y}^1) &\leq 0, \\ \mu(N_{C/Y}^1/N_{C/Y}^0) &\leq \frac{1}{2} \deg T_C(-\log S_C), \\ \mu(N_{C/Y}^0) &\leq \deg T_C(-\log S_C).\end{aligned}$$

Adding all three inequalities we obtain a single inequality

$$\deg N_{C/Y} \leq \frac{\text{rank}(N_{C/Y}^1) + \text{rank}(N_{C/Y}^0)}{2} \cdot \deg T_C(-\log S_C).$$

In case of equality, we say that (RPC) holds.

EXAMPLE 4.2. In case Y is a smooth projective surface, and C is a smooth special curve in Y intersecting the boundary S_Y transversally, then

$$(K_Y + S_Y) \cdot C + 2C^2 = 0,$$

if Y is a Hilbert modular surface, and

$$(K_Y + S_Y) \cdot C + 3C^2 = 0,$$

if Y is a ball quotient, see [11, Thm. 0.1], [12, Ex. 1.6] and [4, Chap. 17].

The main consequence of (RPC) is the following:

PROPOSITION 4.3.

(i) If $\varphi : C \hookrightarrow Y$ satisfies (RPC), then $\varphi^*T_Y(-\log S_Y)$ is a direct summand of an orthogonal decomposition of $\varphi^*T_{A_g}(-\log S)$ with respect to the Hodge metric.

(ii) If $Y^0 \hookrightarrow A_g$ is a special subvariety, then $\varphi^*T_Y(-\log S_Y)$ is a direct summand of an orthogonal decomposition of $\varphi^*T_{A_g}(-\log S)$ with respect to the Hodge metric and $\varphi : C \hookrightarrow Y$ satisfies (RPC).

Proof. [12, Prop. 1.5]. □

In [12, Formula 1.3] we showed that, if C^0 is a special curve, one has a splitting

$$\varphi^*T_Y(-\log S_Y) \cong T_C(-\log S_C) \oplus N_{C/Y}.$$

This splitting is induced from a corresponding splitting of $\varphi^*T_{A_g}(-\log S)$. If, in addition, (RPC) holds, then this splitting is compatible with the decomposition

$$N_{C/Y} = \bigoplus_{i=0}^2 N_{C/Y}^i/N_{C/Y}^{i-1}.$$

5 SPECIAL SUBVARIETIES IN $X = \Gamma \backslash D$

As far as we know, there is no good notion of Hecke operators on Mumford-Tate domains $D = G(\mathbb{R})/V$. In addition, there are no good compactifications of $X = \Gamma \backslash D$ known in these cases, since X does not even carry any algebraic structure in general [8]. Here Γ is the monodromy group.

Therefore, to avoid these two difficulties, by a *special curve* in X we will denote an étale morphism

$$\varphi^0 : C^0 \longrightarrow X = \Gamma \backslash G(\mathbb{R})/V$$

from a Shimura curve C^0 to X , induced from a morphism of algebraic groups $G' \rightarrow G$ defined over \mathbb{Q} . In other words, C^0 is the quotient of the orbit of a certain Hodge structure $h \in D$ under the conjugation action of G' . One can generalize this notion to arbitrary special subvarieties.

Let $h \in D$ be a Hodge structure with Mumford-Tate group M_h . The $M_h(\mathbb{R})$ -orbit of h under the conjugation action of $M_h(\mathbb{R})$ in D is a Mumford-Tate domain $D(M_h) \subset D$ in the sense of [7, 4]. In other words, $D(M_h)$ is a connected component of the image of a Shimura datum (M, X') in the Mumford-Tate datum (G, X) , see [4, Chap. 17]. Using this notation, we define:

DEFINITION 5.1. A *special subvariety of Shimura type* in X is a horizontal, algebraic subvariety $Z^0 \subset X$, such that there is a Hodge structure $h \in D$ with Mumford-Tate group $M = M_h$ such that Z^0 is a quotient of the orbit $D(M)$ by a discrete subgroup.

Note that we shall omit reference to the Hodge structure h and simply speak of a Mumford-Tate domain $D(M)$ as in the above definition.

Hence, by our definition of a special subvariety Z^0 , we have a commutative diagram

$$\begin{array}{ccc} D(M) & \longrightarrow & D \\ \downarrow & & \downarrow \\ Z^0 & \longrightarrow & X \end{array}$$

Note that we always require a special subvariety to be horizontal and algebraic, so that Z^0 is of Shimura type, i.e., $D(M)$ is a Hermitian symmetric domain. In most cases, Z^0 is a proper subvariety of X by [8].

REMARK 5.2. More general notions of special subvarieties in Mumford-Tate varieties are conceivable, for example horizontal subvarieties of maximal dimension in Mumford-Tate varieties. But it is not clear whether such definitions have good properties. For example such varieties may not carry any CM points.

6 RELATIVE PROPORTIONALITY IN $X = \Gamma \backslash D$

To define the relative proportionality condition (RPC), using the notation of the previous paragraph, we need first the following observations.

Let $C^0 \xrightarrow{\varphi^0} Y^0 \xrightarrow{i} X$ be a special curve and Y^0 an algebraic subvariety of $X = \Gamma \backslash D$. Let Y be a smooth compactification of Y^0 and C a compatible smooth compactification of C^0 , which extends to a finite morphism $\varphi : C \rightarrow Y$. Note that for this we do not need to require that X has an algebraic compactification. Denote by $S_Y = Y \setminus Y^0$ the boundary divisor, and by $S_C = C \setminus C^0$, so that S_C is the pullback of S_Y to C .

Fix a base point corresponding to a Hodge representation $h : \mathbb{S} \rightarrow G_{\mathbb{R}}$ whose orbit under G defines X . Let $\mathfrak{g} = \text{Lie}(G)$. As mentioned earlier, $\mathfrak{g} = \bigoplus_p \mathfrak{g}^{-p,p}$ carries a weight zero Hodge structure. If K is a maximal compact subgroup containing V , then its complexified Lie algebra $\mathfrak{k}_{\mathbb{C}}$ is given by the sum for even p , whereas its complement $\mathfrak{p}_{\mathbb{C}}$ is the sum for all odd p [4, Sec. 12.5]. For $p = 1$, we obtain the horizontal, holomorphic tangent bundle, see Definition 1.7. The vertical tangent bundle is given by the quotient of Lie algebras $\mathfrak{k}/\mathfrak{v}$, where \mathfrak{v} is the Lie algebra of V . This terminology comes from the fibration [4, 8]

$$\omega : D = G(\mathbb{R})/V \longrightarrow G(\mathbb{R})/K.$$

We remark that the horizontal tangent bundle agrees with the usual tangent bundle, if $V = K$ and D is a Hermitian symmetric domain, for example in the case of $X = A_g$.

PROPOSITION 6.1. *Assume that Y^0 (and hence C^0) has unipotent monodromies at infinity. Then the bundle $(\varphi^0)^*T_X^h$ on C^0 extends to a vector bundle on C which we denote by $\varphi^*T_X^h(-\log S)$.*

Proof. Let $\mathcal{V}^{p,q}$ be the universal vector bundles on D which parametrize the (p, q) -classes on X . The horizontal, holomorphic tangent bundle T_X^h of X is contained in a direct sum of the Hodge bundles:

$$T_X^h \subset \mathcal{E}nd^{-1,1} \left(\bigoplus_{p,q} \mathcal{V}^{p,q} \right) = \bigoplus_{p,q} \mathcal{H}om(\mathcal{V}^{p,q}, \mathcal{V}^{p-1,q+1}).$$

All these bundles are homogenous on D , and the inclusion of the subbundle T_X is defined by explicit conditions. Over the algebraic variety Y^0 , the restricted bundles $\mathcal{V}^{p,q}|_{Y^0}$ on the right hand side, and also the subbundle $T_X|_{Y^0}$, have a Deligne extension $\overline{\mathcal{V}}^{p,q}|_{Y^0}$ to Y . Therefore, $T_X^h|_{Y^0}$ and $(\varphi^0)^*T_X^h$ have natural extensions to Y and C which we denote by $\varphi^*T_X^h(-\log S)$, although S does not exist. □

Using this, we can define the logarithmic normal bundle $N_{C/X}$ through the exact sequence

$$0 \rightarrow T_C(-\log S_C) \rightarrow \varphi^*T_X^h(-\log S) \rightarrow N_{C/X} \rightarrow 0.$$

In a similar way, we have the exact sequence

$$0 \rightarrow T_C(-\log S_C) \rightarrow \varphi^* T_Y(-\log S_Y) \rightarrow N_{C/Y} \rightarrow 0.$$

By a previous result [12, Prop. 1.5.(ii)] of ours, see Prop. 4.3(ii) above, which is independent of A_g , we know that the logarithmic tangent bundle $T_C(-\log S_C)$ is an orthogonal direct summand of the newly defined bundle $\varphi^* T_X(-\log S)$ with respect to the Hodge metric:

$$T_C(-\log S_C) \hookrightarrow \varphi^* T_X(-\log S).$$

We now show that certain local systems on C^0 split in a controlled way, giving a representation-theoretic proof of the following result of [17].

LEMMA 6.2. *Assume that \mathbb{V} is a \mathbb{C} -variation of Hodge structures of weight k over C^0 which comes from a $G(\mathbb{R})$ -representation on X by restriction. Then,*

$$\mathbb{V} = \mathbb{U} \oplus \bigoplus_i (S^i(\mathbb{L}) \otimes \mathbb{T}_i),$$

where \mathbb{L} is a weight one local system of rank 2 and \mathbb{T}_i and \mathbb{U} are unitary local systems of weights $k - i$ and k respectively.

Proof. Since C^0 splits in at least one place, we may assume that the special Mumford-Tate group of the Shimura curve C^0 has the form $SL(2) \times U_1 \times \dots \times U_r$ for some $r \geq 0$, where the U_i are compact Lie groups (i.e., anisotropic). This gives rise to an embedding $SL(2) \times U_1 \times \dots \times U_r \hookrightarrow G$ of algebraic groups. Now, since the groups U_i are compact as real groups, it follows that every representation of them is a unitary representation and it is well-known that the representations of the group $SL(2)_{\mathbb{R}}$ are direct sums of symmetric products of the standard representation. Note also that the irreducible subrepresentations of the product representation is a product of the irreducible subrepresentations of each representation and that the product of unitary representations is again unitary. This means that there is a standard 2-dimensional representation \mathbb{L} and unitary representations \mathbb{T}_i and \mathbb{U} such that \mathbb{V} has the asserted decomposition. \square

Note that, since \mathbb{L} is a weight 1 variation of Hodge structures, and C^0 is a special curve, by results of [17], its Deligne extension to C corresponds to a Higgs bundle of the form $(\mathcal{L} \oplus \mathcal{L}^{-1}, \sigma)$ such that the Higgs field $\sigma : \mathcal{L} \rightarrow \mathcal{L}^{-1} \otimes \Omega_C^1(\log S_C)$ is an isomorphism, and hence $\mathcal{L}^2 \simeq \Omega_C^1(\log S_C)$.

We can now apply Lemma 6.2 to the universal \mathbb{C} -local system $\mathbb{V}_{\mathbb{C}}$ of weight $k \geq 1$ on X coming from the \mathbb{Q} -local system \mathbb{v} described above. It implies that $(\varphi^0)^* \mathbb{V}_{\mathbb{C}} = \mathbb{U} \oplus \bigoplus_i (S^i(\mathbb{L}) \otimes \mathbb{T}_i)$ for local systems \mathbb{L} , \mathbb{T}_i and \mathbb{U} over C^0 . We denote the Higgs bundles on C corresponding to the \mathbb{C} -local systems $(\varphi^0)^* \mathbb{V}_{\mathbb{C}}$, \mathbb{T}_i and \mathbb{U} by \mathcal{V} , \mathcal{T}_i and \mathcal{U} . The bundles \mathcal{T}_i and \mathcal{U} have degree 0 and their Higgs fields are zero. Note that the Higgs field of $S^i(\mathbb{L})$ comes from that of \mathcal{L} , i.e.,

is equal to $S^i(\sigma)$ for σ the Higgs field of \mathcal{L} . The Higgs field of \mathbb{V} respects the direct sums and vanishes on \mathbb{U} . Therefore,

$$T_C(-\log S_C) \subseteq \bigoplus_{\square} \mathcal{H}om(\mathcal{L}^{i-2\mu} \otimes \mathcal{T}_{i,a}, \mathcal{L}^{j-2\nu} \otimes \mathcal{T}_{j,b}) \subset \mathcal{E}nd^{-1,1} \left(\bigoplus_{p+q=k} \mathcal{V}^{p,q} \right),$$

where the bundles $\mathcal{T}_{i,a}, \mathcal{T}_{j,b}$ have slope 0, and $\square = \{(\mu, i, \nu, j, a, b) \in \mathbb{N}_0^6 \mid \mu \leq i \leq k, \nu \leq j \leq k, a \leq k-i, b \leq k-j, j+b-\nu = i+a-\mu-1\}$. In the above sum, $T_C(-\log S_C)$ is a direct summand and orthogonal with respect to the natural Riemannian (i.e., Hodge) metric. Let $T_C(-\log S_C)^\perp$ denote the orthogonal complement of $T_C(-\log S_C)$ in this sum. Thus, there is a decomposition

$$\varphi^*T_X(-\log S) = T_C(-\log S_C) \oplus N_{C/X},$$

such that, as in [12, Section 1],

$$N_{C/X} \subset T_C(-\log S_C)^\perp \oplus \bigoplus_{p+q=k} \mathcal{H}om(\mathcal{U}^{p,q}, \mathcal{V}^{p-1,q+1}) \oplus \bigoplus_{p+q=k} \mathcal{H}om(\mathcal{V}^{p,q}, \mathcal{U}^{p-1,q+1}).$$

In particular, $N_{C/X}$ is a sum of polystable bundles of different slopes. Hence, one has a Harder-Narasimhan decomposition

$$N_{C/X} = \bigoplus_{i=0}^s R_i$$

with polystable bundles R_i of strictly increasing slopes $\mu(R_i) < \mu(R_{i+1})$. The length s is an integer depending on C and X .

Accordingly, the Harder-Narasimhan filtration on $N_{C/X}$ is given by

$$N_{C/X}^i = R_0 \oplus R_1 \oplus \dots \oplus R_i, \quad 0 \leq i \leq s.$$

Taking the induced filtration $N_{C/Y}^i := N_{C/X}^i \cap N_{C/Y}$ on $N_{C/Y}$ obtained by intersection, we get a filtration on $N_{C/Y}$:

$$N_{C/Y}^0 \subset N_{C/Y}^1 \subset \dots \subset N_{C/Y}^s = N_{C/Y}.$$

In analogy with the A_g case, we can now make the following definition:

DEFINITION 6.3 (Relative Proportionality Condition (RPC)).

We say that $\varphi : C \rightarrow Y$ satisfies the *relative proportionality condition* (RPC), if the slope inequalities

$$\mu(N_{C/Y}^i/N_{C/Y}^{i-1}) \leq \mu(N_{C/X}^i/N_{C/X}^{i-1}), \quad i = 0, \dots, s$$

are equalities.

Adding all these inequalities, we obtain a single inequality

$$\deg N_{C/Y} \leq r(C, Y, X) \cdot \deg T_C(-\log S_C),$$

where $r(C, Y, X) \in \mathbb{Q}$ is a rational number depending on C, Y and X , and hence on G . However, it is not possible to write $r(C, Y, X)$ in a closed form, as in the case of $X = A_g$, since it would depend on G and not only on the weight k . In example 4.2, the constant $r(C, Y, X)$ is 1 in the case of Hilbert modular surfaces and $\frac{1}{2}$ in the case of ball quotients. The assertions of Proposition 4.3 and [12, Formula 1.3] also hold in this more general case by induction over s , i.e., we have a splitting

$$\varphi^* T_Y(-\log S_Y) \cong T_C(-\log S_C) \oplus \bigoplus_{i=0}^s N_{C/Y}^i / N_{C/Y}^{i-1},$$

if C^0 is a special curve in X satisfying (RPC).

7 PROOF OF THEOREM 1.11

In this section we prove Theorem 1.11. From this, Theorem 1.5 and Theorem 1.9 follow, as we showed in the introduction.

We assume conditions (BIG) and (RPC) and look at a smooth and horizontal subvariety $Y^0 \hookrightarrow X$, where $X = \Gamma \backslash G(\mathbb{R})/K$ is a (connected) Mumford-Tate variety.

We choose a base point $y \in \bigcup_i C_i^0$. Note that X carries a family of \mathbb{Q} -Hodge structures \mathbb{V} as a local system. It does not underly a variation of \mathbb{Q} -Hodge structures in general, since Griffiths transversality may not hold. However, when restricted to Y^0 , or the curves C_i^0 , this will be the case, since Y^0 is horizontal. We now consider the restriction of \mathbb{V} to Y^0 only.

Let H be the \mathbb{Q} -algebraic group from condition (LIE). By Definition 1.8, it fixes precisely the Hodge classes in these vector spaces $W_{y \in Y}$. For ease of notation, set $C = C_i$ in what follows (till end of the proof of Lemma 7.1) and denote the restriction $\mathbb{V}_\mathbb{C}$ to C by \mathbb{V}_C . Since the \mathbb{C} -local system \mathbb{V}_C has the form $\mathbb{V}_C = \bigoplus (S^i(\mathbb{L}) \otimes \mathbb{T}_i) \oplus \mathbb{U}$, where \mathbb{L} is related to a \mathbb{C} -local system of weight 1 corresponding to a Higgs bundle $\mathcal{L} \oplus \mathcal{L}^{-1}$, by Lemma 6.2 and (proof of) [17], Proposition 3.4, the Higgs field is given by

$$S^{i-2\mu}(\sigma) : \mathcal{L}^{i-2\mu} \rightarrow \mathcal{L}^{i-2\mu-2} \otimes \Omega_Y^1(\log S).$$

In particular, the sheaves $E^{p,q}$ can be decomposed into a direct sum of polystable sheaves $E_\iota^{p,q}$ of slopes $\mu(E_\iota^{p,q}) = \iota \deg \mathcal{L}$ for $\iota \in [-qk, \dots, pk]$. Using this, we prove:

LEMMA 7.1. *The thickening $\theta_{y \in Y}$ on $E_\iota^{p,q}$ decomposes as a direct sum of morphisms:*

$$E_\iota^{p,q} \xrightarrow{\theta_{N_{C/Y}^i / N_{C/Y}^{i-1}}} E_{\iota+r_i}^{p-1,q+1} \otimes (N_{C/Y}^i / N_{C/Y}^{i-1})^\vee$$

between polystable bundles of the same slope. Here, r_i is the number satisfying $\mu(R_i) = \mu(N_{C/Y}^i/N_{C/Y}^{i-1}) = r_i \deg \mathcal{L}$.

Proof. Note that the above decomposition of $N_{C/Y}$ gives a corresponding decomposition as

$$\theta_C + \theta_{N_{C/Y}} = \theta_C + \theta_{N_{C/Y}^0} + \theta_{N_{C/Y}^1/N_{C/Y}^0} + \cdots + \theta_{N_{C/Y}^s/N_{C/Y}^{s-1}}.$$

Since, by Lemma 6.2 on the curve C , one has the \mathbb{C} -local system $\mathbb{V}_C = \bigoplus (S^i(\mathbb{L}) \otimes \mathbb{T}_i) \oplus \mathbb{U}$, the description of the sheaves $E^{p,q}$ shows that we can reduce to the situation $i = 1$. This case is treated in [12, Lemma 2.7] for $\mathbb{V}_C^{\otimes k}$. In fact, if $i = 1$, then for $k = 1$ we have the decompositions

$$\begin{aligned} \mathcal{L} \otimes \mathcal{T} &\rightarrow \mathcal{L}^{-1} \otimes \mathcal{T} \otimes \Omega_C^1(\log S_C) \\ \mathcal{L} \otimes \mathcal{T} &\rightarrow \mathcal{L}^{-1} \otimes \mathcal{T} \otimes (N_{C/Y}^0)^\vee \\ \mathcal{L} \otimes \mathcal{T} &\rightarrow \mathcal{U}^\vee \otimes (N_{C/Y}^1/N_{C/Y}^0)^\vee \\ \mathcal{U} &\rightarrow \mathcal{U}^\vee \otimes (N_{C/Y}^2/N_{C/Y}^1)^\vee \end{aligned}$$

and for arbitrary weight k , the result can be obtained by reducing to the case $k = 1$ by remembering that $\theta_{y \in Y}^{\otimes k}$ is defined by the Leibniz rule. □

Thus, we have shown that the kernels of ϑ decompose into vector bundles with vanishing slopes, and hence induce unitary Higgs bundles. This is the crucial ingredient for the remaining proof.

PROPOSITION 7.2. *Under condition (RPC), the subspaces $W_{y \in Y, \mathbb{Q}}$ and $W_{y \in Y}$ are invariant under the monodromy action of $\pi_1(\bigcup_i C_i^0, y) \rightarrow G$ and define a unitary local system on each curve C_i^0 . If conditions (RPC) and (BIG) both hold, then the subspaces $W_{y \in Y, \mathbb{Q}}$ and $W_{y \in Y}$ are invariant under the monodromy action of $\pi_1(Y^0, y) \rightarrow G$.*

Proof. The proofs of Prop. 2.4 and Prop. 3.1 of [12] immediately carry over to this more general situation. Indeed the kernels $\ker(\theta^{p,p})$ decompose to subbundles with non-positive slopes and one shows that $\ker(\theta^{p,p})_0$, the subbundle of slope 0, is invariant under complex conjugation induced by the real structure and underlies a unipotent subsystem \mathbb{W} of type (p, p) . Conversely, any $t \in W_{y \in Y, \mathbb{R}}$ lies in $\ker(\theta^{p,p})_{0,y}$ showing that $\mathbb{W}_{\mathbb{R},y} = W_{y \in Y, \mathbb{R}}$. Proposition 3.1 of [12] says that the subspace $W_{y \in Y, \mathbb{R}}$ is invariant under monodromy action. Fixing a base point $y_1 \in C_1^0$ the idea is to study the parallel transport of real vectors in $W_{y_1 \in Y, \mathbb{R}}$ along paths in the connected subspace $\bigcup_i C_i$. The Higgs field $\theta_{y \in Y}$ decomposes as $\theta_{y \in C_1} + \theta_{N_{C_1/Y}, y}$ and hence $W_{y \in Y, \mathbb{R}}$ can be identified with the kernel of $\theta_{N_{C_1/Y}, y}$. On the other hand, as indicated above, there is a unitary subsystem \mathbb{W}_{C_1} of type (p, p) such that $W_{y \in Y, \mathbb{R}} = \mathbb{W}_{C_1, y}$. The kernel $\theta_{N_{C_1/Y}}$ is a polystable subbundle of slope zero and hence it underlies a unitary subsystem \mathbb{W}'_{C_1} . It follows that $W_{y \in Y, \mathbb{R}} \subset \mathbb{W}'_{C_1, y}$. Now starting with a real

vector $t_1 \in W_{y \in Y, \mathbb{R}}$ and parallel transporting it along some path in C_1^0 from y_1 to $y_2 \in C_1^0 \cap C_2^0$ to a vector t_2 , one sees that t_2 is also a real vector and is contained in the fiber of \mathbb{W}'_{C_1} at y_2 . It therefore follows that $\theta_{y \in C_1}(t_2) = 0$ and $\theta_{N_{C_1/Y}, y}(t_2) = 0$, i.e., $t_2 \in W_{y_2 \in Y, \mathbb{R}}$. Regarding $y_2 \in C_2^0$ and repeating the above parallel transport along a path in C_2^0 , from y_2 to $y_3 \in C_2 \cap C_3$ and so on, one concludes that $W_{y \in Y, \mathbb{R}}$ is invariant under the monodromy action. Finally, the last part of the proof there uses only condition (BIG). \square

As in Cor. 3.5 of [12], one gets the following corollary:

COROLLARY 7.3. *The subspaces $W_{y \in Y}$ define a unitary local subsystem $\mathbb{U} \subset \mathbb{W}$ on Y^0 with \mathbb{Q} -structure. The local system \mathbb{U} extends to Y , and has finite monodromy.*

Since we assumed that the monodromies at infinity are unipotent, which always holds after a finite étale cover of Y^0 , this means that \mathbb{U} is trivial, and all its global sections, i.e., all (p, p) -classes inside \mathbb{U} , which are by definition of $W_{y \in Y}$ invariant under H , are also monodromy-invariant. Recall that the Mumford-Tate group $MT(\mathbb{V}_{Y^0})$ is the \mathbb{Q} -algebraic group fixing all parallel Hodge classes for all p [4, Chap. 15]. We obtain therefore:

COROLLARY 7.4. *The infinitesimally fixed Hodge classes in $\mathbb{W}_{\mathbb{Q}}$ over points $y \in Y^0$ are globally monodromy-invariant. Hence, condition (Mon) holds.*

Therefore, Theorem 1.11 is proven.

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